Let U be a neighbourhood of the origin in X. Then there exists $s > 0$ s.t. $B_s(o) \subseteq U$ and so $f(B_s(o)) \subseteq f(U)$. By applying (1.15) for $r = \frac{s}{2}$, we obtain that $\exists\, \rho := \rho_{\frac{s}{2}} > 0$ s.t. $B_\rho(o) \subseteq \overline{f(B_{\frac{s}{2}}(o))}$ and so, by Lemma 1.5.5, we have $B_\rho(o) \subseteq \overline{f(B_s(o))} \subseteq f(U)$ since $s > \frac{s}{2}$. Hence, $f(U)$ is a neighbourhood of the origin in Y. ☐

Lemma 1.5.4. *Let X be a t.v.s., Y a Baire t.v.s. and $f\colon X \to Y$ a continuous linear map. If $f(X)$ has non-empty interior, then $\overline{f(V)}$ is a neighbourhood of the origin in Y whenever V is a neighbourhood of the origin in X.*

Proof. (see Exercise Sheet 1) ☐

Lemma 1.5.5. *Let X be a metrizable and complete t.v.s. and Y a metrizable (not necessarily complete) t.v.s.. If $f\colon X \to Y$ is a continuous linear map such that*

$$\forall r > 0, \exists\, \rho_r > 0 \;\; s.t. \;\; B_{\rho_r}(f(x)) \subseteq \overline{f(B_r(x))}, \forall x \in X, \tag{1.15}$$

then for any $a > r$ we have that $B_{\rho_r}(f(x)) \subseteq f(B_a(x))$ for all $x \in X$.

Proof. Fixed $a > r > 0$, we can write $a = \sum_{n=0}^{\infty} r_n$ with $r_0 := r$ and $r_n > 0$ for all $n \in \mathbb{N}$. By assumption (1.15), we have that

$$\exists\, \rho_0 := \rho_{r_0} = \rho_r > 0 \;\; \text{s.t.} \;\; B_{\rho_0}(f(x)) \subseteq \overline{f(B_{r_0}(x))}, \forall x \in X, \tag{1.16}$$

and

$$\forall n \in \mathbb{N}, \exists\, \rho_n := \rho_{r_n} > 0 \;\; \text{s.t.} \;\; B_{\rho_n}(f(x)) \subseteq \overline{f(B_{r_n}(x))}, \forall x \in X. \tag{1.17}$$

W.l.o.g. we can assume that $(\rho_n)_{n\in\mathbb{N}}$ is strictly decreasing and convergent to zero.

Let $x \in X$ and $y \in B_{\rho_r}(f(x))$. We want to show that there exists a point $x' \in B_a(x)$ such that $y = f(x')$. To do that, we shall construct a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $(f(x_n))_{n\in\mathbb{N}}$ converges to y. Since X is complete, this will imply that $(x_n)_{n\in\mathbb{N}}$ converges to a point $x' \in X$, which necessarily satisfies $f(x') = y$ as f is continuous and Y Hausdorff. Of course, we need to define the sequence $(x_n)_{n\in\mathbb{N}}$ in such a way that its limit point x' lies in $B_a(x)$.

Since $y \in B_{\rho_r}(f(x)) \overset{(1.16)}{\subseteq} \overline{f(B_{r_0}(x))}$, there exists $x_1 \in B_{r_0}(x)$ such that $d_Y(f(x_1), y) < \rho_1$. Then $y \in B_{\rho_1}(f(x_1)) \overset{(1.17)}{\subseteq} \overline{f(B_{r_1}(x_1))}$ and so there exists

$x_2 \in B_{r_1}(x_1)$ such that $d_Y(f(x_2), y) < \rho_2$, i.e. $y \in B_{\rho_2}(f(x_2))$. By repeatedly applying (1.17), we get that for any $n \in \mathbb{N}$ there exists $x_{n+1} \in B_{r_n}(x_n)$ such that $d_Y(f(x_{n+1}), y) < \rho_{n+1}$. The sequence $(x_n)_{n\in\mathbb{N}}$ so obtained has all the desired properties. Indeed, for any $n \in \mathbb{N}$, we have $d_X(x_n, x_{n+1}) < r_n$ and so for any $m \geq l$ in $\mathbb{N}$ we get $d_X(x_l, x_m) \leq \sum_{j=l}^{m-1} d_X(x_j, x_{j+1}) < \sum_{j=l}^{\infty} r_j$, which implies that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Hence, by the completeness of X, there exists $x' \in X$ such that $d_X(x_n, x) \to 0$ as $n \to \infty$ and for any $n \in \mathbb{N}$ we get that

$$d_X(x, x') \leq d_X(x, x_1) + d_X(x_1, x_2) + \cdots + d_X(x_n, x') < \sum_{j=0}^{n-1} r_j + d_X(x_n, x').$$

Hence, $d_X(x, x') \leq \sum_{j=0}^{\infty} r_j = a$. Furthermore, for any $n \in \mathbb{N}$, we have $0 \leq d_Y(f(x_n), y) < \rho_n \to 0$, which implies the convergence of $(f(x_n))_{n\in\mathbb{N}}$ to y in Y. $\square$

The Open Mapping Theorem 1.5.3 has several applications.

Corollary 1.5.6. *A bijective continuous linear map between two metrizable and complete t.v.s. is a topological isomorphism.*

Proof. Let X and Y be two metrizable and complete t.v.s. and $f : X \to Y$ bijective continuous and linear. Then, by the Open Mapping Theorem 1.5.3, we know that f is open, i.e. for any $U \in \mathcal{F}_X(o)$ we have that $f(U) \in \mathcal{F}_Y(o)$. This means that the inverse f^{-1}, whose existence is ensured by the injectivity of f, is continuous. $\square$

Corollary 1.5.7. *A bijective linear map between two metrizable and complete t.v.s. with continuous inverse is continuous and so a topological isomorphism.*

Proof. Apply Corollary 1.5.6 to the inverse. $\square$

Corollary 1.5.8. *Let τ_1 and τ_2 be two topologies on the same vector space X, both turning X into a metrizable complete t.v.s.. If τ_1 and τ_2 are comparable, then they coincide.*

Proof. Suppose that τ_1 is finer than τ_2. Then the identity map from (X, τ_1) to (X, τ_2) is bijective continuous and linear and so a topological isomorphism by Corollary 1.5.6. This means that also its inverse is continuous, i.e. the identity map from (X, τ_2) to (X, τ_1) is continuous. Hence, τ_2 is finer than τ_1. $\square$

Corollary 1.5.9. *Let p and q be two norms on the same vector space X. If both (X,p) and (X,q) are Banach spaces and there exists $C>0$ such that $p(x)\leq Cq(x)$ for all $x\in X$, then the norms p and q are equivalent.*

Proof. Apply Corollary 1.5.8 to the topologies generated by p and by q. □

A fundamental result which can be derived from the Open Mapping Theorem 1.5.3 is the so called Closed Graph Theorem.

Theorem 1.5.10 (Closed Graph Theorem)**.**
Let X and Y be two metrizable and complete t.v.s.. Every linear map $f: X\to Y$ with closed graph is continuous.

Recall that the graph of a map $f: X\to Y$ is defined by

$$\mathrm{Gr}(f):=\{(x,y)\in X\times Y : y=f(x)\}.$$

The Closed Graph Theorem 1.5.10 will follow at once from the Open Mapping Theorem 1.5.3 and the following general result.

Proposition 1.5.11.
Let X and Y be two t.v.s. such that the following property holds.

$$\text{If } G \text{ is a closed linear subspace of } X\times Y \text{ and } g: G\to X \text{ is a continuous linear surjective map then } g \text{ is open.} \tag{1.18}$$

Then every linear map $f: X\to Y$ with closed graph is continuous.

Proof. Since X and Y are both t.v.s., $X\times Y$ endowed with the product topology is a t.v.s. and so the first and the second coordinate projections are both continuous. As $f: X\to Y$ is linear, $\mathrm{Gr}(f)$ is a linear subspace of $X\times Y$. Hence, $\mathrm{Gr}(f)$ endowed with subspace topology induced by the product topology, is itself a t.v.s. and the coordinate projections restricted to $\mathrm{Gr}(f)$, i.e.

$$\begin{array}{rlcl} p: & \mathrm{Gr}(f) & \to & X \\ & (x,f(x)) & \mapsto & x \end{array} \quad \text{and} \quad \begin{array}{rlcl} q: & \mathrm{Gr}(f) & \to & Y \\ & (x,f(x)) & \mapsto & f(x), \end{array}$$

are both continuous. Moreover, p is also linear and bijective, so there exists its inverse p^{-1} and we have that $f=q\circ p^{-1}$. Since p is a linear bijective and continuous map, (1.18) ensures that p is open, i.e. p^{-1} is continuous. Hence, f is continuous as composition of continuous maps. □

Proof. of Closed Graph Theorem
Since X and Y are both metrizable and complete t.v.s., (1.18) immediately follows from the Open Mapping Theorem 1.5.3. Indeed, if G is a closed linear subspace of $X \times Y$, then G endowed with the subspace topology induced by the product topology is also a metrizable and complete t.v.s. and so any $g : G \to X$ linear continuous and surjective is open by the Open Mapping Theorem 1.5.3. As (1.18) holds, we can apply Proposition 1.5.11, which ensures that every linear map $f : X \to Y$ with $\mathrm{Gr}(f)$ closed is continuous. □

The Closed Graph Theorem 1.5.10 and the Open Mapping Theorem 1.5.3 are actually equivalent, in the sense that we can also derive Theorem 1.5.3 from Theorem 1.5.10. To this purpose, we need to show a general topological result.

Proposition 1.5.12. *Let X and Y be two topological spaces such that Y is Hausdorff. Every continuous map from X to Y has closed graph.*

Proof. Let $f : X \to Y$ be continuous. We want to show that $(X \times Y) \backslash \mathrm{Gr}(f) := \{(x, y) \in X \times Y : y \neq f(x)\}$is open, i.e. for any $(x, y) \in (X \times Y) \backslash \mathrm{Gr}(f)$ we want to show that there exists a neighbourhood W of x in X and a neighbourhood U of y in Y such that $(x, y) \in W \times U \subseteq (X \times Y) \setminus \mathrm{Gr}(f)$.

As Y is Hausdorff and $y \neq f(x)$, there exist U neighbourhood of y in Y and V neighbourhood of $f(x)$ in Y such that $U \cap V = \emptyset$. The continuity of f guarantees that $f^{-1}(V)$ is a neighbourhood of x in X and so we have that $(x, y) \in f^{-1}(V) \times U$. We claim that $f^{-1}(V) \times U \subseteq (X \times Y) \setminus \mathrm{Gr}(f)$. If this was not the case, then there would exist $(\tilde{x}, \tilde{y}) \in f^{-1}(V) \times U$ such that $(\tilde{x}, \tilde{y}) \notin (X \times Y) \backslash \mathrm{Gr}(f)$. Hence, $\tilde{y} = f(\tilde{x}) \in f(f^{-1}(V)) \subseteq V$ and so $\tilde{y} \in U \cap V$ which yields a contradiction. □

Proof. of Open Mapping Theorem 1.5.3 using Closed Graph Theorem 1.5.10
Let f be a linear continuous and surjective map between two metrizable and complete t.v.s. X and Y. Then the map $\bar{f} : X/\mathrm{Ker}(f) \to Y$ defined in (1.14) is linear bijective and continuous by Proposition 1.5.1. Then Proposition 1.5.12 implies that $\mathrm{Gr}(\bar{f})$ is closed in $X/\mathrm{Ker}(f) \times Y$ endowed with the product topology. This gives in turn that the graph $\mathrm{Gr}(\bar{f}^{-1})$ of the inverse of $\bar{f}$ is closed in $Y \times X/\mathrm{Ker}(f)$, as $\mathrm{Gr}(\bar{f}^{-1}) = j(\mathrm{Gr}(\bar{f})$ where $j : X/\mathrm{Ker}(f) \times Y \to Y \times X/\mathrm{Ker}(f)$ is the homeomorphism given by $j(a, b) = (b, a)$. Hence, $\bar{f}^{-1}$ is a linear map with closed graph and so it is continuous by the Closed Graph Theorem 1.5.10. This means that $\bar{f}$ is open and so for any U neighbourhood of the origin in X we have $f(U) = \bar{f}(\phi(U))$ is open, i.e. f is open. □

The main advantage of the Closed Graph Theorem is that in many situations it is easier to prove that the graph of a map is closed rather than showing its continuity directly. For instance, we have seen that the inverse of an injective linear function with closed graph has also closed graph or that the inverse of a linear injective continuous map with Hausdorff codomain has closed graph. In both these cases, when we are in the realm of metrizable and complete t.v.s., we can conclude the continuity of the inverse thanks to the Closed Graph Theorem.