Chapter 2

Bounded subsets of topological vector spaces

In this chapter we will study the notion of bounded set in any t.v.s. and analyzing some properties which will be useful in the following and especially in relation with duality theory. Since compactness plays an important role in the theory of bounded sets, we will start this chapter by recalling some basic definitions and properties of compact subsets of a t.v.s..

2.1 Preliminaries on compactness

Let us recall some basic definitions of compact subset of a topological space (not necessarily a t.v.s.)

Definition 2.1.1. *A topological space X is said to be* compact *if X is Hausdorff and if every open covering $\{\Omega_i\}_{i\in I}$ of X contains a finite subcovering, i.e. for any collection $\{\Omega_i\}_{i\in I}$ of open subsets of X s.t. $\bigcup_{i\in I}\Omega_i = X$ there exists a finite subset $J \subseteq I$ s.t. $\bigcup_{j\in J}\Omega_j = X$.*

By going to the complements, we obtain the following equivalent definition of compactness.

Definition 2.1.2. *A topological space X is said to be* compact *if X is Hausdorff and if every family $\{F_i\}_{i\in I}$ of closed subsets of X whose intersection is empty contains a finite subfamily whose intersection is also empty, i.e. for any collection $\{F_i\}_{i\in I}$ of closed subsets of X s.t. $\bigcap_{i\in I} F_i = \emptyset$ there exists a finite subset $J \subseteq I$ s.t. $\bigcap_{j\in J} F_j = \emptyset$.*

Definition 2.1.3. *A subset K of a topological space X is said to be* compact *if K endowed with the topology induced by X is Hausdorff and for any collection $\{\Omega_i\}_{i\in I}$ of open subsets of X s.t. $\bigcup_{i\in I}\Omega_i \supseteq K$ there exists a finite subset $J \subseteq I$ s.t. $\bigcup_{j\in J}\Omega_j \supseteq K$.*

Let us state without proof a few well-known properties of compact spaces.

Proposition 2.1.4.

a) A closed subset of a compact space is compact.

b) Finite unions of compact sets are compact.

c) Let f be a continuous mapping of a compact space X into a Hausdorff topological space Y. Then $f(X)$ is a compact subset of Y.

In the following we will almost always be concerned with compact subsets of a Hausdorff t.v.s. X carrying the topology induced by X (and so which are themselves Hausdorff t.v.s.). Therefore, we now introduce a useful characterization of compactness in Hausdorff topological spaces.

Theorem 2.1.5. *Let X be a Hausdorff topological space. X is compact if and only if every filter on X has at least one accumulation point (see Definition 1.3.8).*

Proof.
Suppose that X is compact. Let $\mathcal{F}$ be a filter on X and $\mathcal{C} := \{\overline{M} : M \in \mathcal{F}\}$. As $\mathcal{F}$ is a filter, no finite intersection of elements in $\mathcal{C}$ can be empty. Therefore, by compactness, the intersection of all elements in $\mathcal{C}$ cannot be empty. Then there exists at least a point $x \in \overline{M}$ for all $M \in \mathcal{F}$, i.e. x is an accumulation point of $\mathcal{F}$. Conversely, suppose that every filter on X has at least one accumulation point. Let ϕ be a family of closed subsets of X whose intersection is empty. To show that X is compact, we need to show that there exists a finite subfamily of ϕ whose intersection is empty. Suppose by contradiction that no finite subfamily of ϕ has empty intersection. Then the family ϕ' of all the finite intersections of subsets belonging to ϕ forms a basis of a filter $\mathcal{F}$ on X. By our initial assumption, $\mathcal{F}$ has an accumulation point, say x. Thus, x belongs to the closure of any element of $\mathcal{F}$ and in particular to any set belonging to ϕ' (as the elements in ϕ' clearly belong to $\mathcal{F}$ and are closed). This means that x belongs to the intersection of all the sets belonging to ϕ', which is the same as the intersection of all the sets belonging to ϕ. But we had assumed the latter to be empty and so we have a contradiction. □

Corollary 2.1.6.
Any compact subset of a Hausdorff topological space is closed.

Proof.
Let K be a compact subset of a Hausdorff topological space X and let $x \in \overline{K}$. Denote by $\mathcal{F}(x)$ the filter of neighbourhoods of x in X and by $\mathcal{F}(x) \upharpoonright K$ the filter in K generated by all the sets $U \cap K$ where $U \in \mathcal{F}(x)$. By Theorem 2.1.5, $\mathcal{F}(x) \upharpoonright K$ has an accumulation point $x_1 \in K$. We claim that $x_1 \equiv x$, which implies that $\overline{K} = K$ and so that K is closed. In fact, if $x_1 \neq x$ then

the Hausdorffness of X implies that there exists $U \in \mathcal{F}(x)$ s.t. $X \setminus U$ is a neighbourhood of x_1 and, thus, $x_1 \notin \overline{U \cap K}$, which contradicts the fact that x_1 is an accumulation point of $\mathcal{F}(x) \upharpoonright K$. □

Corollary 2.1.7.

1) Arbitrary intersections of compact subsets of a Hausdorff topological space are compact.

2) Any bijective continuous map from a compact space to a Hausdorff space is a homeomorphism.

3) Let τ_1, τ_2 be two Hausdorff topologies on a set X. If $\tau_1 \subseteq \tau_2$ and (X, τ_2) is compact then $\tau_1 \equiv \tau_2$.

Proof.

1. Let X be a Hausdorff topological space and $\{K_i\}_{i\in I}$ be an arbitrary family of compact subsets of X. Then each K_i is closed in X by Corollary 2.1.6 and so $\bigcap_{i\in I} K_i$ is a closed subset of each fixed K_i. As K_i is compact, Proposition 2.1.4-a) ensures that $\bigcap_{i\in I} K_i$ is compact in K_i and so in X.
2. Let U be an open subset of a compact space X and f a continuous map from X to a Hausdorff space Y. Since $X \setminus U$ is closed in X and X is compact, we have that $X \setminus U$ is compact in X by Proposition 2.1.4-a). Then Proposition 2.1.4-c) guarantees that $f(X \setminus U)$ is compact in Y, which implies in turn that $f(X \setminus U)$ is closed in Y by Corollary 2.1.6. Since f is bijective, we have that $Y \setminus f(U) = f(X \setminus U)$ and so that $f(U)$ is open. Hence, f^{-1} is continuous.
3. Since $\tau_1 \subseteq \tau_2$, the identity map from (X, τ_2) to (X, τ_1) is continuous and clearly bijective. Then the previous item implies that the identity from (X, τ_1) to (X, τ_2) is also continuous. Hence, $\tau_1 \equiv \tau_2$.

□

Last but not least, let us recall the following two definitions.

Definition 2.1.8. *A subset A of a topological space X is said to be* relatively compact *if the closure $\overline{A}$ of A is compact in X.*

Definition 2.1.9. *A subset A of a Hausdorff t.v.s. E is said to be precompact if A is relatively compact when viewed as a subset of the completion $\hat{E}$ of E.*

2.2 Bounded subsets: definition and general properties

Definition 2.2.1. *A subset B of a t.v.s. E is said to be* bounded *if for every U neighbourhood of the origin in E there exists $\lambda > 0$ such that $B \subseteq \lambda U$.*

In rough words this means that a subset B of E is bounded if B can be swallowed by any neighbourhood of the origin.

Proposition 2.2.2.

1. *If every element in some basis of neighbourhoods of the origin of a t.v.s. swallows a subset, then such a subset is bounded.*
2. *The closure of a bounded set is bounded.*
3. *Finite unions of bounded sets are bounded sets.*
4. *Any subset of a bounded set is a bounded set.*

Proof. Let E be a t.v.s. and $B \subset E$.

1. Suppose that $\mathcal{N}$ is a basis of neighbourhoods of the origin o in E such that for every $N \in \mathcal{N}$ there exists $\lambda_N > 0$ with $B \subseteq \lambda_N N$. Then, by definition of basis of neighbourhoods of o, for every U neighbourhood of o in E there exists $M \in \mathcal{N}$ s.t. $M \subseteq U$. Hence, there exists $\lambda_M > 0$ s.t. $B \subseteq \lambda_M M \subseteq \lambda_M U$, i.e. B is bounded.
2. Suppose that B is bounded in E. Then, as there always exists a basis $\mathcal{C}$ of neighbourhoods of the origin in E consisting of closed sets (see Corollary 2.1.14-a) in TVS-I), we have that for any $C \in \mathcal{C}$ there exists $\lambda > 0$ s.t. $B \subseteq \lambda C$ and thus $\overline{B} \subseteq \overline{\lambda C} = \lambda \overline{C} = \lambda C$. By Proposition 2.2.2-1, this is enough to conclude that $\overline{B}$ is bounded in E.
3. Let $n \in \mathbb{N}$ and $B_1, \ldots, B_n$ bounded subsets of E. As there always exists a basis $\mathcal{B}$ of balanced neighbourhoods of the origin in E (see Corollary 2.1.14-b) in TVS-I), we have that for any $V \in \mathcal{B}$ there exist $\lambda_1, \ldots, \lambda_n > 0$ s.t. $B_i \subseteq \lambda_i V$ for all $i = 1, \ldots, n$. Then $\bigcup_{i=1}^n B_i \subseteq \bigcup_{i=1}^n \lambda_i V \subseteq \left(\max_{i=1,\ldots,n} \lambda_i\right) V$, which implies the boundedness of $\bigcup_{i=1}^n B_i$ by Proposition 2.2.2-1.
4. Let B be bounded in E and let A be a subset of B. The boundedness of B guarantees that for any neighbourhood U of the origin in E there exists $\lambda > 0$ s.t. λU contains B and so A. Hence, A is bounded.

□

The properties in Proposition 2.2.2 lead to the following definition which is dually corresponding to the notion of basis of neighbourhoods.

Definition 2.2.3. *Let E be a t.v.s.. A family $\{B_\alpha\}_{\alpha\in I}$ of bounded subsets of E is called a* basis of bounded subsets *of E if for every bounded subset B of E there is $\alpha \in I$ s.t. $B \subseteq B_\alpha$.*

This duality between neighbourhoods and bounded subsets will play an important role in the study of the strong topology on the dual of a t.v.s.

Which sets do we know to be bounded in any t.v.s.?
- Singletons are bounded in any t.v.s., as every neighbourhood of the origin is absorbing.
- Finite subsets in any t.v.s. are bounded as finite union of singletons.

Proposition 2.2.4. *Compact subsets of a t.v.s. are bounded.*

Proof. Let E be a t.v.s. and K be a compact subset of E. For any neighbourhood U of the origin in E we can always find an open and balanced neighbourhood V of the origin s.t. $V \subseteq U$. Then we have

$$K \subseteq E = \bigcup_{n=0}^{\infty} nV.$$

From the compactness of K, it follows that there exist finitely many integers $n_1, \dots, n_r \in \mathbb{N}_0$ s.t.

$$K \subseteq \bigcup_{i=1}^{r} n_i V \subseteq \left(\max_{i=1,\dots,r} n_i\right) V \subseteq \left(\max_{i=1,\dots,r} n_i\right) U.$$

Hence, K is bounded in E. □

This together with Corollary 2.1.6 gives that in any Hausdorff t.v.s. a compact subset is always bounded and closed. In finite dimensional Hausdorff t.v.s. we know that also the converse holds (because of Theorem 3.1.1 in TVS-I) and thus the ***Heine-Borel property*** always holds, i.e.

$$K \text{ compact} \Leftrightarrow K \text{ bounded and closed.}$$

This is not true, in general, in infinite dimensional t.v.s.