Definition 2.2.3. *Let E be a t.v.s.. A family $\{B_\alpha\}_{\alpha\in I}$ of bounded subsets of E is called a* basis of bounded subsets *of E if for every bounded subset B of E there is $\alpha \in I$ s.t. $B \subseteq B_\alpha$.*

This duality between neighbourhoods and bounded subsets will play an important role in the study of the strong topology on the dual of a t.v.s.

Which sets do we know to be bounded in any t.v.s.?
- Singletons are bounded in any t.v.s., as every neighbourhood of the origin is absorbing.
- Finite subsets in any t.v.s. are bounded as finite union of singletons.

Proposition 2.2.4. *Compact subsets of a t.v.s. are bounded.*

Proof. Let E be a t.v.s. and K be a compact subset of E. For any neighbourhood U of the origin in E we can always find an open and balanced neighbourhood V of the origin s.t. $V \subseteq U$. Then we have

$$K \subseteq E = \bigcup_{n=0}^{\infty} nV.$$

From the compactness of K, it follows that there exist finitely many integers $n_1, \ldots, n_r \in \mathbb{N}_0$ s.t.

$$K \subseteq \bigcup_{i=1}^{r} n_i V \subseteq \left(\max_{i=1,\ldots,r} n_i\right) V \subseteq \left(\max_{i=1,\ldots,r} n_i\right) U.$$

Hence, K is bounded in E. □

This together with Corollary 2.1.6 gives that in any Hausdorff t.v.s. a compact subset is always bounded and closed. In finite dimensional Hausdorff t.v.s. we know that also the converse holds (because of Theorem 3.1.1 in TVS-I) and thus the ***Heine-Borel property*** always holds, i.e.

$$K \text{ compact} \Leftrightarrow K \text{ bounded and closed.}$$

This is not true, in general, in infinite dimensional t.v.s.

Example 2.2.5.
Let E be an infinite dimensional normed space. If every bounded and closed subset in E were compact, then in particular all the balls centered at the origin would be compact. Then the space E would be locally compact and so finite dimensional as proved in Theorem 3.2.1 in TVS-I, which gives a contradiction.

There is however an important class of infinite dimensional t.v.s., the so-called *Montel spaces*, in which the Heine-Borel property holds. Note that $\mathcal{C}^\infty(\mathbb{R}^d), \mathcal{C}_c^\infty(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d)$ are all Montel spaces.

Proposition 2.2.4 provides some further interesting classes of bounded subsets in a Hausdorff t.v.s..

Corollary 2.2.6. *Precompact subsets of a Hausdorff t.v.s. are bounded.*

Proof.
Let K be a precompact subset of a Hausdorff t.v.s. E and i the canonical embedding of E in its completion $\hat{E}$. By Definition 2.1.9, the closure of $i(K)$ in $\hat{E}$ is compact. Let U be any neighbourhood of the origin in E. Since i is a topological embedding, there is a neighbourhood $\hat{U}$ of the origin in $\hat{E}$ such that $U = \hat{U} \cap E$. Then, by Proposition 2.2.4, there is a number $\lambda > 0$ such that $\overline{i(K)} \subseteq \lambda \hat{U}$. Hence, we get

$$K \subseteq \overline{i(K)} \cap E \subseteq \lambda \hat{U} \cap E = \lambda \hat{U} \cap \lambda E = \lambda(\hat{U} \cap E) = \lambda U.$$ □

Corollary 2.2.7. *Let E be a Hausdorff t.v.s.. The union of a converging sequence in E and of its limit is a compact and so bounded closed subset in E.*

Proof. (Christmas assignment) □

Corollary 2.2.8. *Let E be a Hausdorff t.v.s.. Any Cauchy sequence in E is bounded.*

Proof. By using Corollary 2.2.7, one can show that any Cauchy sequence S in E is a precompact subset of E. Then it follows by Corollary 2.2.6 that S is bounded in E. □

Note that a Cauchy sequence S in a Hausdorff t.v.s. E is not necessarily relatively compact in E. Indeed, if this were the case, then its closure in E would be compact and so, by Theorem 2.1.5, the filter associated to S would have an accumulation point $x \in E$. Hence, by Proposition 1.3.10 and Proposition 1.1.31 in TVS-I, we get $S \to x \in E$ which is not necessarily true unless E is complete.

Proposition 2.2.9. *The image of a bounded set under a continuous linear map between t.v.s. is a bounded set.*

Proof. Let E and F be two t.v.s., $f : E \to F$ be linear and continuous, and $B \subseteq E$ be bounded. Then for any neighbourhood V of the origin in F, $f^{-1}(V)$ is a neighbourhood of the origin in E. By the boundedness of B in E, it follows that there exists $\lambda > 0$ s.t. $B \subseteq \lambda f^{-1}(V)$ and, thus, $f(B) \subseteq \lambda V$. Hence, $f(B)$ is a bounded subset of F. □

Corollary 2.2.10. *Let L be a continuous linear functional on a t.v.s. E. If B is a bounded subset of E, then $\sup_{x \in B} |L(x)| < \infty$.*

Proof. By Proposition 2.2.9, we have that $L(B)$ is bounded in $\mathbb{K}$. Hence, there exists $\lambda > 0$ such that $L(B)$ is contained in the closed ball of radius λ centered at the origin, i.e. for all $x \in B$ we have $|L(x)| \leq \lambda$, which yields the conclusion. □

Let us now introduce a general characterization of bounded sets in terms of sequences.

Proposition 2.2.11. *Let E be any t.v.s.. A subset B of E is bounded if and only if every sequence contained in B is bounded in E.*

Proof. The necessity of the condition is obvious from Proposition 2.2.2-4. Let us prove its sufficiency. Suppose that every sequence contained in B is bounded in E. If B is unbounded, then there exists a neighbourhood U of the origin in E s.t. for all $\lambda > 0$ we have $B \not\subseteq \lambda U$. W.l.o.g. we can assume U balanced. Then

$$\forall\, n \in \mathbb{N},\ \exists\, x_n \in B \ \text{ s.t. } x_n \notin nU. \tag{2.1}$$

By assumption the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded and so there exists $\mu > 0$ s.t. $\{x_n\}_{n\in\mathbb{N}} \subseteq \mu U$. Hence, there exists $m \in \mathbb{N}$ with $m \geq \mu$ such that $\{x_n\}_{n\in\mathbb{N}} \subseteq mU$ and in particular $x_m \in mU$, which contradicts (2.1). Hence, B must necessarily be bounded in E. □

2.3 Bounded subsets of special classes of t.v.s.

In this section we are going to study bounded sets in some of the special classes of t.v.s. which we have encountered so far. First of all, let us notice that any ball in a normed space is a bounded set and thus that in normed spaces there exist sets which are at the same time bounded and neighbourhoods of the origin. This property is actually a characteristic of all normable Hausdorff locally convex t.v.s.. Recall that a t.v.s. E is said to be *normable* if its topology can be defined by a norm, i.e. if there exists a norm $\|\cdot\|$ on E such that the collection $\{B_r : r > 0\}$ with $B_r := \{x \in E : \|x\| < r\}$ is a basis of neighbourhoods of the origin in E.

Proposition 2.3.1. *Let E be a Hausdorff locally convex t.v.s.. If there is a neighbourhood of the origin in E which is also bounded, then E is normable.*

Proof. Let U be a bounded neighbourhood of the origin in E. As E is locally convex, by Proposition 4.1.12 in TVS-I, we may always assume that U is open and absolutely convex, i.e. convex and balanced. The boundedness of U implies that for any balanced neighbourhood V of the origin in E there exists $\lambda > 0$ s.t. $U \subseteq \lambda V$. Hence, $U \subseteq nV$ for all $n \in \mathbb{N}$ such that $n \geq \lambda$, i.e. $\frac{1}{n}U \subseteq V$. Then the collection $\left\{\frac{1}{n}U\right\}_{n\in\mathbb{N}}$ is a basis of neighbourhoods of the origin o in E and, since E is a Hausdorff t.v.s., Corollary 2.2.4 in TVS-I guarantees that

$$\bigcap_{n\in\mathbb{N}} \frac{1}{n}U = \{o\}. \tag{2.2}$$

Since E is locally convex and U is an open absolutely convex neighbourhood of the origin, there exists a generating seminorm p on E s.t. $U = \{x \in E : p(x) < 1\}$ (see second part of proof of Theorem 4.2.9 in TVS-I). Then p must be a norm, because $p(x) = 0$ implies $x \in \frac{1}{n}U$ for all $n \in \mathbb{N}$ and so $x = 0$ by (2.2). Hence, E is normable. □

An interesting consequence of this result is the following one.

Corollary 2.3.2. *Let E be a locally convex metrizable space. If E is not normable, then E cannot have a countable basis of bounded sets in E.*

Proof. (Exercise Sheet 6) □

The notion of boundedness can be extended to linear maps between t.v.s..

Definition 2.3.3. *Let E, F be two t.v.s. and f a linear map of E into F. f is said to be* bounded *if for every bounded subset B of E, $f(B)$ is a bounded subset of F.*

We have already showed in Proposition 2.2.9 that any continuous linear map between two t.v.s. is a bounded map. The converse is not true in general but it holds for two special classes of t.v.s.: metrizable t.v.s. and LF-spaces.

Proposition 2.3.4. *Let E be a metrizable t.v.s. and let f be a linear map of E into a t.v.s. F. If f is bounded, then f is continuous.*

Proof. Let $f : E \to F$ be a bounded linear map. Suppose that f is not continuous. Then there exists a neighbourhood V of the origin in F whose preimage $f^{-1}(V)$ is not a neighbourhood of the origin in E. W.l.o.g. we can always assume that V is balanced. As E is metrizable, we can take a countable basis $\{U_n\}_{n\in\mathbb{N}}$ of neighbourhood of the origin in E s.t. $U_n \supseteq U_{n+1}$ for all $n \in \mathbb{N}$. Then for all $m \in \mathbb{N}$ we have $\frac{1}{m}U_m \not\subseteq f^{-1}(V)$ i.e.

$$\forall\, m \in \mathbb{N},\, \exists\, x_m \in \frac{1}{m}U_m \text{ s.t. } f(x_m) \notin V. \tag{2.3}$$

As for all $m \in \mathbb{N}$ we have $mx_m \in U_m$ we get that the sequence $\{mx_m\}_{m\in\mathbb{N}}$ converges to the origin o in E. In fact, for any neighbourhood U of the origin o in E there exists $\bar{n} \in \mathbb{N}$ s.t. $U_{\bar{n}} \subseteq U$. Then for all $n \geq \bar{n}$ we have $nx_n \in U_n \subseteq U_{\bar{n}} \subseteq U$, i.e. $\{mx_m\}_{m\in\mathbb{N}}$ converges to o.

Hence, Proposition 2.2.7 implies that $\{mx_m\}_{m\in\mathbb{N}_0}$ is bounded in E and so, since f is bounded, also $\{mf(x_m)\}_{m\in\mathbb{N}_0}$ is bounded in F. This means that there exists $\rho > 0$ s.t. $\{mf(x_m)\}_{m\in\mathbb{N}_0} \subseteq \rho V$. Then for all $n \in \mathbb{N}$ with $n \geq \rho$ we have $f(x_n) \in \frac{\rho}{n}V \subseteq V$ which contradicts (2.3).

□

To show that the previous proposition also hold for LF-spaces, we need to introduce the following characterization of bounded sets in LF-spaces.

Proposition 2.3.5.
Let (E, τ_{ind}) be an LF-space with defining sequence $\{(E_n, \tau_n)\}_{n\in\mathbb{N}}$. A subset B of E is bounded in E if and only if there exists $n \in \mathbb{N}$ s.t. B is contained in E_n and B is bounded in E_n.