Topological Vector Spaces II

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Contents

The primary source for these notes is [5]. However, especially in the presentation of Section 1.3, 1.4 and 3.3, we also followed [4] and [3] are. The references to results from TVS-I (WS 2018/19) appear in the following according to the enumeration used in [2].

Chapter 1

Special classes of topological vector spaces

In these notes we consider vector spaces over the field $\mathbb{K}$ of real or complex numbers given the usual euclidean topology defined by means of the modulus.

1.1 Metrizable topological vector spaces

Definition 1.1.1. *A t.v.s. X is said to be* metrizable *if there exists a metric d which defines the topology of X.*

We recall that a metric d on a set X is a mapping $d : X \times X \to \mathbb{R}^+$ with the following properties:

1. $d(x, y) = 0$ if and only if $x = y$ (identity of indiscernibles);
2. $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetry);
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (triangular inequality).

To say that the topology of a t.v.s. X is defined by a metric d means that for any $x \in X$ the sets of all open (or equivalently closed) balls:

$$B_r(x) := \{y \in X : d(x, y) < r\}, \quad \forall r > 0$$

forms a basis of neighbourhoods of x w.r.t. to the original topology on X.

There exists a completely general characterization of metrizable t.v.s..

Theorem 1.1.2. *A t.v.s. X is metrizable if and only if X is Hausdorff and has a countable basis of neighbourhoods of the origin.*

One direction is quite straightforward. Indeed, suppose that X is a metrizable t.v.s. and that d is a metric defining the topology of X, then the collection of all $B_{\frac{1}{n}}(o)$ with $n \in \mathbb{N}$ is a countable basis of neighbourhoods of the origin o in X. Moreover, the intersection of all these balls is just the singleton $\{o\}$, which proves that the t.v.s. X is also Hausdorff (see Corollary 2.2.4 in TVS-I).

The other direction requires more work and we are not going to prove it in full generality but only for locally convex (l.c.) t.v.s., since this class of t.v.s. is anyway the most commonly used in applications. Before doing it, let us make another general observation:

Proposition 1.1.3. *In any metrizable t.v.s. X, there exists a translation invariant metric which defines the topology of X.*

Recall that a metric d on X is said to be *translation invariant* if

$$d(x+z, y+z) = d(x, y), \qquad \forall\, x, y, z \in X.$$

It is important to highlight that the converse of Proposition 1.1.3 does not hold in general. Indeed, the topology τ_d defined on a vector space X by a translation invariant metric d is a translation invariant topology and also the addition is always continuous w.r.t. τ_d. However, the multiplication by scalars might be not continuous w.r.t. τ_d and so (X, τ_d) is not necessarily a t.v.s.. For example, the discrete metric on any non-trivial vector space X is translation invariant but the discrete topology on X is not compatible with the multiplication by scalars (see Interactive Sheet 1).

Proof. (of Theorem 1.1.2 and Proposition 1.1.3 for l.c. t.v.s.)
Let X be a l.c. t.v.s.. Suppose that X is Hausdorff and has a countable basis $\{U_n, n \in \mathbb{N}\}$ of neighbourhoods of the origin. Since X is a l.c. t.v.s., we can assume that such a countable basis of neighbourhoods of the origin consists of barrels, i.e. closed, convex, absorbing and balanced sets (see Proposition 4.1.13 in TVS-I) and that satisfies the following property (see Theorem 4.1.14 in TVS-I):

$$\forall j \in \mathbb{N}, \forall \rho > 0, \exists\, n \in \mathbb{N} : U_n \subset \rho U_j.$$

We may then take

$$V_n = U_1 \cap \cdots \cap U_n, \quad \forall n \in \mathbb{N}$$

as a basis of neighbourhoods of the origin in X. Each V_n is a still barrel, $V_{n+1} \subseteq V_n$ for any $n \in \mathbb{N}$ and:

$$\forall j \in \mathbb{N}, \forall \rho > 0, \exists\, n \in \mathbb{N} : V_n \subset \rho V_j. \tag{1.1}$$

By Lemma 4.2.7 in TVS-I we know that for any $n \in \mathbb{N}$ we have $V_n \subseteq U_{p_{V_n}}$, where $p_{V_n} := \{\lambda > 0 : x \in \lambda V_n\}$ is the Minkowski functional associated to V_n and $U_{p_{V_n}} := \{x \in X : p_{V_n}(x) \leq 1\}$. Also, if $x \in U_{p_{V_n}}$ then there exists a sequence $(\lambda_j)_{j\in\mathbb{N}}$ such that $\lambda_j > 0$ and $x \in \lambda_j V_n$ for each $j \in \mathbb{N}$, and $\lambda_j \to 1$

as $j \to \infty$. This implies that $\frac{x}{\lambda_j} \to x$ as $j \to \infty$ and so $x \in V_n$ since V_n is closed. Hence, we have just showed that for any $n \in \mathbb{N}$ there is a seminorm p_n (i.e. $p_n := p_{V_n}$) on X such that $V_n = \{x \in X : p_n(x) \leq 1\}$. Then clearly we have that $(p_n)_{n\in\mathbb{N}}$ is a countable family of seminorms generating the topology of X and such that $p_n \leq p_{n+1}$ for all $n \in \mathbb{N}$.

Let us now fix a sequence of real positive numbers $\{a_j\}_{j\in\mathbb{N}}$ such that $\sum_{j=1}^{\infty} a_j < \infty$ and define the mapping d on $X \times X$ as follows:

$$d(x,y) := \sum_{j=1}^{\infty} a_j \frac{p_j(x-y)}{1+p_j(x-y)}, \qquad \forall, x, y \in X.$$

We want to show that this is a metric which defines the topology of X.

Let us immediately observe that the positive homogeneity of the seminorms p_j gives that d is a symmetric function. Also, since X is a Hausdorff t.v.s., we get that $\{o\} \subseteq \cap_{n=1}^{\infty}\mathrm{Ker}(p_n) \subseteq \cap_{n=1}^{\infty} V_n = \{o\}$, i.e. $\cap_{n=1}^{\infty}\mathrm{Ker}(p_n) = \{o\}$. This provides that $d(x,y) = 0$ if and only if $x = y$. We must therefore check the triangular inequality for d. This will follow by applying, for any fixed $j \in \mathbb{N}$ and $x, y, z \in X$, Lemma 1.1.4 below to $a := p_j(x-y)$, $b := p_j(y-z)$ and $c := p_j(x-z)$. In fact, since each p_j is a seminorm on X, we have that the above defined a, b, c are all non-negative real numbers such that: $c = p_j(x-z) = p_j(x-y+y-z) \leq p_j(x-y) + p_j(y-z) = a+b$. Hence, the assumption of Lemma 1.1.4 are fulfilled for such a choice of a, b and c and we get that for each $j \in \mathbb{N}$:

$$\frac{p_j(x-z)}{1+p_j(x-z)} \leq \frac{p_j(x-y)}{1+p_j(x-y)} + \frac{p_j(y-z)}{1+p_j(y-z)}, \qquad \forall x, y, z \in X.$$

Since the a_j's are all positive, this implies that $d(x,z) \leq d(x,y) + d(y,z)$, $\forall x, y, z \in X$. We have then proved that d is indeed a metric and from its definition it is clear that it is also translation invariant.

To complete the proof, we need to show that the topology defined by this metric d coincides with the topology initially given on X. By Hausdorff criterion (see Theorem 1.1.17 in TVS-I), we therefore need to prove that for any $x \in X$ both the following hold:

1. $\forall r > 0, \exists n \in \mathbb{N} : x + V_n \subseteq B_r(x)$
2. $\forall n \in \mathbb{N}, \exists r > 0 : B_r(x) \subseteq x + V_n$

Because of the translation invariance of both topologies, we can consider just the case $x = o$.

Let us fix $r > 0$. As $\sum_{j=1}^{\infty} a_j < \infty$, we can find $j(r) \in \mathbb{N}$ such that

$$\sum_{j=j(r)+1}^{\infty} a_j < \frac{r}{2}. \tag{1.2}$$

Using that $p_n \leq p_{n+1}$ for all $n \in \mathbb{N}$ and denoting by A the sum of the series of the a_j's, we get:

$$\sum_{j=1}^{j(r)} a_j \frac{p_j(x)}{1+p_j(x)} \leq p_{j(r)}(x) \sum_{j=1}^{j(r)} a_j \leq p_{j(r)}(x) \sum_{j=1}^{\infty} a_j = A p_{j(r)}(x). \tag{1.3}$$

Combining (1.2) and (1.3), we get that if $x \in \frac{r}{2A} V_{j(r)}$, i.e. if $p_{j(r)}(x) \leq \frac{r}{2A}$, then:

$$d(x,o) = \sum_{j=1}^{j(r)} a_j \frac{p_j(x)}{1+p_j(x)} + \sum_{j=j(r)+1}^{\infty} a_j \frac{p_j(x)}{1+p_j(x)} < A p_{j(r)}(x) + \frac{r}{2} \leq r.$$

This proves that $\frac{r}{2A} V_{j(r)} \subseteq B_r(o)$. By (1.1), there always exists $n \in \mathbb{N}$ s.t. $V_n \subseteq \frac{r}{2A} V_{j(r)}$ and so 1 holds. To prove 2, let us fix $j \in \mathbb{N}$. Then clearly

$$a_j \frac{p_j(x)}{1+p_j(x)} \leq d(x,o), \qquad \forall x \in X.$$

As the a_j's are all positive, the latter implies that:

$$p_j(x) \leq a_j^{-1}(1+p_j(x)) d(x,o), \qquad \forall x \in X.$$

Therefore, if $x \in B_{\frac{a_j}{2}}(o)$ then $d(x,o) \leq \frac{a_j}{2}$ and so $p_j(x) \leq \frac{(1+p_j(x))}{2}$, which gives $p_j(x) \leq 1$. Hence, $B_{\frac{a_j}{2}}(o) \subseteq V_j$ which proves 2. □

Let us show now the small lemma used in the proof above:

Lemma 1.1.4. *Let $a, b, c \in \mathbb{R}^+$ such that $c \leq a+b$ then $\frac{c}{1+c} \leq \frac{a}{1+a} + \frac{b}{1+b}$.*

Proof. W.l.o.g. we can assume $c > 0$ and $a+b > 0$. (Indeed, if $c = 0$ or $a+b = 0$ then there is nothing to prove.)Then $c \leq a+b$ is equivalent to $\frac{1}{a+b} \leq \frac{1}{c}$. This implies that $\left(1+\frac{1}{c}\right)^{-1} \leq \left(1+\frac{1}{a+b}\right)^{-1}$ which is equivalent to:

$$\frac{c}{1+c} \leq \frac{a+b}{1+a+b} = \frac{a}{1+a+b} + \frac{b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b}.$$

□

We have therefore the following characterization of l.c. metrizable t.v.s.:

Proposition 1.1.5. *A locally convex t.v.s. (X, τ) is metrizable if and only if τ can be generated by a countable separating family of seminorms.*

Let us introduce now three general properties of all metrizable t.v.s. (not necessarily l.c.), which are well-known in the theory of metric spaces.

Proposition 1.1.6. *A metrizable t.v.s. X is complete if and only if X is sequentially complete.*

Proof. (Exercise Sheet 1) □

(For the definitions of completeness and sequentially completeness of a t.v.s., see Definition 2.5.5 and Definition 2.5.6 in TVS-I. See also Proposition 2.5.7 and Example 2.5.11 n TVS-I for more details on the relation between these two notions for general t.v.s..)

Proposition 1.1.7. *Let X be a metrizable t.v.s. and Y be any t.v.s. (not necessarily metrizable). A mapping $f : X \to Y$ (not necessarily linear) is continuous if and only if it is sequentially continuous.*

Proof. (Exercise Sheet 1) □

Recall that a mapping f from a topological space X into a topological space Y is said to be *sequentially continuous* if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ convergent to a point $x \in X$ the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $f(x)$ in Y.

The proof that continuity of $f : X \to Y$ always implies its sequentially continuity is pretty straightforward and holds under the general assumption that X and Y are topological spaces (see Proposition 1.1.39 in TVS-I). The converse does not hold in general as the following example shows.

Example 1.1.8.
Let us consider the set $\mathcal{C}([0,1])$ of all real-valued continuous functions on $[0,1]$. This is a vector space w.r.t. the pointwise addition and multiplication by real scalars. We endow $\mathcal{C}([0,1])$ with two topologies which both make it into a t.v.s.. The first topology σ is the one give by the metric:

$$d(f,g) := \int_0^1 \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}, \quad \forall f,g \in \mathcal{C}([0,1]).$$

The second topology τ is instead the topology generated by the family $(p_x)_{x\in[0,1]}$ of seminorms on $\mathcal{C}([0,1])$, where

$$p_x(f) := |f(x)|, \forall f \in \mathcal{C}([0,1]).$$

We will show that the identity map $I : (\mathcal{C}([0,1]),\tau) \to (\mathcal{C}([0,1]),\sigma)$ is sequentially continuous but not continuous.

• *I is sequentially continuous*
Let $(f_n)_{n\in\mathbb{N}}$ *be a sequence of elements in* $\mathcal{C}([0,1])$ *which is* $\tau-$*convergent to* $f \in \mathcal{C}([0,1])$ *as* $n \to \infty$, *i.e.* $|f_n(x) - f(x)| \to 0$, $\forall x \in [0,1]$ *as* $n \to \infty$. *Set*

$$g_n(x) := \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|}, \quad \forall x \in [0,1], \forall n \in \mathbb{N}.$$

Then $|g_n(x)| \leq 1$, $\forall x \in [0,1]$, $\forall n \in \mathbb{N}$ *and* $g_n(x) \to 0$, $\forall x \in [0,1]$ *as* $n \to \infty$. *Hence, by the Lebesgue dominated convergence theorem, we get* $\int_0^1 g_n(x)dx \to 0$ *as* $n \to \infty$, *that is,* $d(f_n, f) \to 0$ *as* $n \to \infty$, *i.e. the sequence* $(I(f_n))_{n\in\mathbb{N}}$ *is* $\sigma-$*convergent to* f *as* $n \to \infty$.
• *I is not continuous*
Suppose that I *is continuous at* $o \in \mathcal{C}([0,1])$ *and fix* $\varepsilon \in (0,1)$. *Then there exists a neighbourhood* N *of the origin in* $(\mathcal{C}([0,1]), \tau)$ *s.t.* $N \subset I^{-1}(B^d_\varepsilon(o))$, *where* $B^d_\varepsilon(o) := \{f \in \mathcal{C}([0,1]) : d(f,0) \leq \varepsilon\}$. *This means that there exist* $n \in \mathbb{N}$, $x_1, \ldots, x_n \in [0,1]$ *and* $\delta > 0$ *s.t.*

$$\bigcap_{i=1}^{n} \delta U_{p_{x_i}} \subset B^d_\varepsilon(o), \tag{1.4}$$

where $U_{p_{x_i}} := \{f \in \mathcal{C}([0,1]) : |f(x_i)| \leq 1\}$.
Take now $f_k(x) := k(x - x_1)\cdots(x - x_n)$, $\forall k \in \mathbb{N}, \forall x \in [0,1]$. *Then* $f_k \in \mathcal{C}([0,1])$ *for all* $k \in \mathbb{N}$ *and* $f_k(x_i) = 0 < \delta$ *for all* $i = 1, \ldots, n$. *Hence,*

$$f_k \in \bigcap_{i=1}^{n}\{f \in \mathcal{C}([0,1]) : |f(x_i)| \leq \delta\} = \bigcap_{i=1}^{n} \delta U_{p_{x_i}} \overset{(1.4)}{\subset} B^d_\varepsilon(o), \ \forall k \in \mathbb{N} \tag{1.5}$$

Set

$$h_k(x) := \frac{|f_k(x)|}{1 + |f_k(x)|}, \quad \forall x \in [0,1], \forall k \in \mathbb{N}.$$

Then $|h_k(x)| \leq 1$, $\forall x \in [0,1]$, $\forall k \in \mathbb{N}$ *and* $h_k(x) \to 1 \, \forall x \in [0,1] \setminus \{x_1, \ldots, x_n\}$ *as* $k \to \infty$. *Hence, by the Lebesgue dominated convergence theorem, we get* $\int_0^1 h_k(x)dx \to \int_0^1 1dx = 1$ *as* $k \to \infty$, *that is,* $d(f_k, f) \to 1$ *as* $k \to \infty$. *This together with (1.5) gives tha* $\varepsilon \geq 1$ *which contradicts our assumption* $\varepsilon \in (0,1)$.

By Proposition 1.1.7, we then conclude that $(\mathcal{C}([0,1]), \tau)$ *is not metrizable.*

Proposition 1.1.9. *A complete metrizable t.v.s.* X *is a Baire space, i.e.* X *fulfills any of the following properties:*
(B) the union of any countable family of closed sets, none of which has interior points, has no interior points.
(B') the intersection of any countable family of everywhere dense open sets is an everywhere dense set.

Note that the equivalence of (B) and (B') is easily given by taking the complements. Indeed, the complement of a closed set C without interior points is clearly open and we get: $X \setminus (\overline{X \setminus C}) = \mathring{C} = \emptyset$ which is equivalent to $\overline{X \setminus C} = X$, i.e. $X \setminus C$ is everywhere dense.

Example 1.1.10. *An example of Baire space is $\mathbb{R}$ with the euclidean topology. Instead $\mathbb{Q}$ with the subset topology given by the euclidean topology on $\mathbb{R}$ is not a Baire space. Indeed, for any $q \in \mathbb{Q}$ the subset $\{q\}$ is closed and has empty interior in $\mathbb{Q}$, but $\cup_{q\in\mathbb{Q}}\{q\} = \mathbb{Q}$ which has interior points in $\mathbb{Q}$ (actually its interior is the whole $\mathbb{Q}$).*

Before proving Proposition 1.1.9, let us observe that the converse of the proposition does not hold because there exist Baire spaces which are not metrizable. Moreover, the assumptions of Proposition 1.1.9 cannot be weakened, because there exist complete non-metrizable t.v.s and metrizable non-complete t.v.s which are not Baire spaces.

Proof. of Proposition 1.1.9
We are going to prove that Property (B') holds in any complete metrizable t.v.s.. Let $\{\Omega_k\}_{k\in\mathbb{N}}$ be a sequence of dense open subsets of X and let us denote by A their intersection. We need to show that A intersects every non-empty open subset of X (this implies that A is dense, since every neighbourhood of every point in X contains some open set and hence some point of A).

Let O be an arbitrary non-empty open subset of X. Since X is a metrizable t.v.s., there exists a countable basis $\{U_k\}_{k\in\mathbb{N}}$ of neighbourhoods of the origin which we may take all closed and s.t. $U_{k+1} \subseteq U_k$ for all $k \in \mathbb{N}$. As Ω_1 is open and dense we have that $O \cap \Omega_1$ is open and non-empty. Therefore, there exists $x_1 \in O \cap \Omega_1$ and $k_1 \in \mathbb{N}$ s.t. $x_1 + U_{k_1} \subseteq O \cap \Omega_1$. Let us denote by G_1 the interior of $x_1 + U_{k_1}$, which is non-empty since it contains x_1 (Indeed, $x_1 + U_{k_1}$ is a neighbourhood of x_1 and so there exists an open set V such that $x_1 \in V \subset x_1 + U_{k_1}$, i.e. x_1 belongs to the interior of $x_1 + U_{k_1}$).

As Ω_2 is dense and G_1 is a non-empty open set, we have that $G_1 \cap \Omega_2$ is open and non-empty. Hence, there exists $x_2 \in G_1 \cap \Omega_2$ and $k_2 \in \mathbb{N}$ s.t. $x_2 + U_{k_2} \subseteq G_1 \cap \Omega_2$. Let us choose $k_2 > k_1$ and call G_2 the interior of $x_2 + U_{k_2}$, which is non-empty since it contains x_2. Proceeding in this way, we get a sequence of non-empty open sets $\mathcal{G} := \{G_l\}_{l\in\mathbb{N}}$ with the following properties for any $l \in \mathbb{N}$:

1. $\overline{G_l} \subseteq \Omega_l \cap O$
2. $G_{l+1} \subseteq G_l$
3. $G_l \subseteq x_l + U_{k_l}$.

Note that the family $\mathcal{G}$ does not contain the empty set and Property 2 implies that for any $G_j, G_k \in \mathcal{G}$ the intersection $G_j \cap G_k = G_{\max\{j,k\}} \in \mathcal{G}$. Hence, $\mathcal{G}$ is a basis of a filter $\mathcal{F}$ in X[1]. Moreover, Property 3 implies that

$$\forall l \in \mathbb{N}, G_l - G_l \subseteq U_{k_l} - U_{k_l} \tag{1.6}$$

which guarantees that $\mathcal{F}$ is a Cauchy filter in X. Indeed, for any neighbourhood U of the origin in X there exists a balanced neighbourhood of the origin such that $V - V \subseteq U$ and so there exists $k \in \mathbb{N}$ such that $U_k \subseteq V$. Hence, there exists $l \in \mathbb{N}$ s.t. $k_l \geq l$ and so $U_{k_l} \subseteq U_k$. Then by (1.6) we have that $G_l - G_l \subseteq U_{k_l} - U_{k_l} \subseteq V - V \subseteq U$. Since $G_l \in \mathcal{G}$ and so in $\mathcal{F}$, we have got that $\mathcal{F}$ is a Cauchy filter.

As X is complete, the Cauchy filter $\mathcal{F}$ has a limit point $x \in X$, i.e. the filter of neighbourhoods of x is contained in the filter $\mathcal{F}$. This implies that $x \in \overline{G_l}$ for all $l \in \mathbb{N}$ (If there would exists $l \in \mathbb{N}$ s.t. $x \notin \overline{G_l}$ then there would exists a neighbourhood N of x s.t. $N \cap G_l = \emptyset$. As $G_l \in \mathcal{F}$ and any neighbourhood of x belongs to $\mathcal{F}$, we get $\emptyset \in \mathcal{F}$ which contradicts the definition of filter.) Hence:

$$x \in \bigcap_{l \in \mathbb{N}} \overline{G_l} \subseteq O \cap \bigcap_{l \in \mathbb{N}} \Omega_l = O \cap A.$$

□

[1] Recall that a basis of a filter on X is a family $\mathcal{G}$ of non-empty subsets of X such that $\forall G_1, G_2 \in \mathcal{G}, \exists G_3 \in \mathcal{G}$ s.t. $G_3 \subset G_1 \cap G_2$.

1.2 Fréchet spaces

Definition 1.2.1. *A complete metrizable locally convex t.v.s. is called a* Fréchet space *(or* F-space*).*

Note that by Theorem 1.1.2 and Proposition 1.1.9, any Fréchet space is in particular a Hausdorff Baire space. Combining the properties of metrizable t.v.s. which we proved in Exercise Sheet 1 and the results about complete t.v.s. which we have seen in TVS-I, we easily get the following properties:

- Any closed linear subspace of an F-space endowed with the induced subspace topology is an F-space.
- The product of a countable family of F-spaces endowed with the product topology is an F-space.
- The quotient of an F-space modulo a closed subspace endowed with the quotient topology is an F-space.

Examples of F-spaces are: Hausdorff finite dimensional t.v.s., Hilbert spaces, and Banach spaces. In the following we will present two examples of F-spaces which do not belong to any of these categories. Let us first recall some standard notations. For any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, we define $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. For any $\beta \in \mathbb{N}_0^d$, the symbol D^β denotes the partial derivative of order $|\beta|$ where $|\beta| := \sum_{i=1}^d \beta_i$, i.e.

$$D^\beta := \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}} = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_d}}{\partial x_d^{\beta_d}}.$$

Example: $\mathcal{C}^s(\Omega)$ with $\Omega \subseteq \mathbb{R}^d$ open.

Let $\Omega \subseteq \mathbb{R}^d$ open in the euclidean topology. For any $s \in \mathbb{N}_0$, we denote by $\mathcal{C}^s(\Omega)$ the set of all real valued $s-$*times continuously differentiable functions* on Ω, i.e. all the derivatives of order $\leq s$ exist (at every point of Ω) and are continuous functions in Ω. Clearly, when $s = 0$ we get the set $\mathcal{C}(\Omega)$ of all real valued continuous functions on Ω and when $s = \infty$ we get the so-called set of all *infinitely differentiable functions* or *smooth functions* on Ω. For any $s \in \mathbb{N}_0$, $\mathcal{C}^s(\Omega)$ (with pointwise addition and scalar multiplication) is a vector space over $\mathbb{R}$.

Let us consider the following family $\mathcal{P}$ of seminorms on $\mathcal{C}^s(\Omega)$

$$p_{m,K}(f) := \sup_{\substack{\beta \in \mathbb{N}_0^d \\ |\beta| \leq m}} \sup_{x \in K} \left| (D^\beta f)(x) \right|, \forall K \subset \Omega \text{ compact}, \forall m \in \{0, 1, \ldots, s\}.$$

(Note when $s = \infty$ we have $m \in \mathbb{N}_0$.) The topology $\tau_\mathcal{P}$ generated by $\mathcal{P}$ is usually referred as $\mathcal{C}^s$*-topology* or *topology of uniform convergence on compact sets of the functions and their derivatives up to order* s.

1) The $\mathcal{C}^s$-topology clearly turns $\mathcal{C}^s(\Omega)$ into a locally convex t.v.s., which is evidently Hausdorff as the family $\mathcal{P}$ is separating (see Prop 4.3.3 in TVS-I). Indeed, if $p_{m,K}(f) = 0$, $\forall m \in \{0, 1, \dots, s\}$ and $\forall K$ compact subset of Ω then in particular $p_{0,\{x\}}(f) = |f(x)| = 0$ $\forall x \in \Omega$, which implies $f \equiv 0$ on Ω.

2) $(\mathcal{C}^s(\Omega), \tau_{\mathcal{P}})$ is metrizable.
By Proposition 1.1.5, this is equivalent to prove that the $\mathcal{C}^s$-topology can be generated by a countable separating family of seminorms. In order to show this, let us first observe that for any two non-negative integers $m_1 \leq m_2 \leq s$ and any two compact $K_1 \subseteq K_2 \subset \Omega$ we have:

$$p_{m_1,K_1}(f) \leq p_{m_2,K_2}(f), \quad \forall f \in \mathcal{C}^s(\Omega).$$

Then the family $\{p_{s,K} : K \subset \Omega \text{ compact}\}$ generates the $\mathcal{C}^s-$topology on $\mathcal{C}^s(\Omega)$. Moreover, it is easy to show that there is a sequence of compact subsets $\{K_j\}_{j\in\mathbb{N}}$ of Ω such that $K_j \subseteq \mathring{K}_{j+1}$ for all $j \in \mathbb{N}$ and $\Omega = \cup_{j\in\mathbb{N}} K_j$. Then for any $K \subset \Omega$ compact we have that there exists $j \in \mathbb{N}$ s.t. $K \subseteq K_j$ and so $p_{s,K}(f) \leq p_{s,K_j}(f)$, $\forall f \in \mathcal{C}^s(\Omega)$. Hence, the countable family of seminorms $\{p_{s,K_j} : j \in \mathbb{N}\}$ generates the $\mathcal{C}^s-$topology on $\mathcal{C}^s(\Omega)$ and it is separating. Indeed, if $p_{s,K_j}(f) = 0$ for all $j \in \mathbb{N}$ then for every $x \in \Omega$ we have $x \in K_i$ for some $i \in \mathbb{N}$ and so $0 \leq |f(x)| \leq p_{s,K_i}(f) = 0$, which implies $|f(x)| = 0$ for all $x \in \Omega$, i.e. $f \equiv 0$ on Ω.

3) $(\mathcal{C}^s(\Omega), \tau_{\mathcal{P}})$ is complete.
By Proposition 1.1.6, it is enough to show that it is sequentially complete. Let $(f_\nu)_{\nu\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{C}^k(\Omega)$, i.e.

$$\forall m \leq s, \forall K \subset \Omega \text{ compact}, \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall \mu, \nu \geq N : p_{m,K}(f_\nu - f_\mu) \leq \varepsilon. \tag{1.7}$$

In particular, for any $x \in \Omega$ by taking $m = 0$ and $K = \{x\}$ we get that the sequence $(f_\nu(x))_{\nu\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Hence, by the completeness of $\mathbb{R}$, it has a limit point in $\mathbb{R}$ which we denote by $f(x)$. Obviously $x \mapsto f(x)$ is a function on Ω, so we have just showed that the sequence $(f_\nu)_{\nu\in\mathbb{N}}$ converges to f pointwise in Ω, i.e.

$$\forall x \in \Omega, \forall \varepsilon > 0, \exists M_x \in \mathbb{N} \text{ s.t. } \forall \mu \geq M_x : |f_\mu(x) - f(x)| \leq \varepsilon. \tag{1.8}$$

Then it is easy to see that $(f_\nu)_{\nu\in\mathbb{N}}$ converges uniformly to f in every compact subset K of Ω. Indeed, we get it just passing to the pointwise limit for $\mu \to \infty$ in (1.7) for $m = 0$. [2]

[2] Detailed proof: Let $\varepsilon > 0$. By (1.7) for $m = 0$, $\exists N \in \mathbb{N}$ s.t.$\forall \mu, \nu \geq N : |f_\nu(x) - f_\mu(x)| \leq \frac{\varepsilon}{2}, \forall x \in K$. Now for each fixed $x \in K$ one can always choose a μ_x larger than both N and the corresponding M_x as in (1.8) so that $|f_{\mu_x}(x) - f(x)| \leq \frac{\varepsilon}{2}$. Hence, for all $\nu \geq N$ one gets that $|f_\nu(x) - f(x)| \leq |f_\nu(x) - f_{\mu_x}(x)| + |f_{\mu_x}(x) - f(x)| \leq \varepsilon, \forall x \in K$.

As $(f_\nu)_{\nu\in\mathbb{N}}$ converges uniformly to f in every compact subset K of Ω, by taking this subset identical with a suitable neighbourhood of any point of Ω, we conclude by Lemma 1.2.2 that f is continuous in Ω.

- If $s=0$, this completes the proof since we just showed $f_\nu \to f$ in the $\mathcal{C}^0-$topology and $f \in \mathcal{C}(\Omega)$.
- If $0<s<\infty$, then observe that since $(f_\nu)_{\nu\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}^s(\Omega)$, for each $j\in\{1,\dots,d\}$ the sequence $(\frac{\partial}{\partial x_j} f_\nu)_{\nu\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}^{s-1}(\Omega)$. Then proceeding as above we can conclude that, for each $j\in\{1,\dots,d\}$, the sequence $(\frac{\partial}{\partial x_j} f_\nu)_{\nu\in\mathbb{N}}$ converges uniformly on every compact subset of Ω to a function $g^{(j)}\in\mathcal{C}^{s-1}(\Omega)$ and by Lemma 1.2.3 we have that $g^{(j)}=\frac{\partial}{\partial x_j} f$. Hence, by induction on s, we show that $(f_\nu)_{\nu\in\mathbb{N}}$ converges to $f\in\mathcal{C}^s(\Omega)$ in the $\mathcal{C}^s-$topology.
- If $s=\infty$, then we are also done by the definition of the $\mathcal{C}^\infty$-topology. Indeed, a Cauchy sequence $(f_\nu)_{\nu\in\mathbb{N}}$ in $\mathcal{C}^\infty(\Omega)$ it is in particular a Cauchy sequence in the subspace topology given by $\mathcal{C}^s(\Omega)$ for any $s\in\mathbb{N}$ and hence, for what we have already showed, it converges to $f\in\mathcal{C}^s(\Omega)$ in the C^s-topology for any $s\in\mathbb{N}$. This means exactly that $(f_\nu)_{\nu\in\mathbb{N}}$ converges to $f\in\mathcal{C}^\infty(\Omega)$ in the in $\mathcal{C}^\infty-$topology.

Let us prove now the two lemmas which we have used in the previous proof:

Lemma 1.2.2. *Let $A\subset\mathbb{R}^d$ and $(f_\nu)_{\nu\in\mathbb{N}}$ in $\mathcal{C}(A)$. If $(f_\nu)_{\nu\in\mathbb{N}}$ converges to a function f uniformly in A then $f\in\mathcal{C}(A)$.*

Proof.
Let $x_0\in A$ and $\varepsilon>0$. By the uniform convergence of $(f_\nu)_{\nu\in\mathbb{N}}$ to f in A we get that:

$$\exists N\in\mathbb{N} \text{ s.t. } \forall\nu\geq N: |f_\nu(y)-f(y)|\leq\frac{\varepsilon}{3}, \forall y\in A.$$

Fix such a ν. As f_ν is continuous on A, we obtain that

$$\exists\delta>0 \text{ s.t. } \forall x\in A \text{ with } |x-x_0|\leq\delta \text{ we have } |f_\nu(x)-f_\nu(x_0)|\leq\frac{\varepsilon}{3}.$$

Therefore, $\forall x\in A$ with $|x-x_0|\leq\delta$ we get

$$|f(x)-f(x_0)|\leq|f(x)-f_\nu(x)|+|f_\nu(x)-f_\nu(x_0)|+|f_\nu(x_0)-f(x_0)|\leq\varepsilon.$$

□

Lemma 1.2.3. *Let $A \subset \mathbb{R}^d$ and $(f_\nu)_{\nu\in\mathbb{N}}$ in $\mathcal{C}^1(A)$. If $(f_\nu)_{\nu\in\mathbb{N}}$ converges to a function f uniformly in A and for each $j \in \{1,\dots,d\}$ the sequence $(\frac{\partial}{\partial x_j} f_\nu)_{\nu\in\mathbb{N}}$ converges to a function $g^{(j)}$ uniformly in A, then*

$$g^{(j)} = \frac{\partial}{\partial x_j} f,\ \forall j \in \{1,\dots,d\}.$$

This means in particular that $f \in \mathcal{C}^1(A)$.

Proof. (for $d = 1$, $A = [a, b]$)
By the fundamental theorem of calculus, we have that for any $x \in A$

$$f_\nu(x) - f_\nu(a) = \int_a^x \frac{\partial}{\partial t} f_\nu(t)dt. \tag{1.9}$$

By the uniform convergence of the first derivatives to $g^{(1)}$ and by the Lebesgue dominated convergence theorem, we also have

$$\int_a^x \frac{\partial}{\partial t} f_\nu(t)dt \to \int_a^x g^{(1)}(t)dt, \text{ as } \nu \to \infty. \tag{1.10}$$

Using (1.9) and (1.10) together with the assumption that $f_\nu \to f$ unformly in A, we obtain that:

$$f(x) - f(a) = \int_a^x g^{(1)}(t)dt,$$

i.e. $\left(\frac{\partial}{\partial x} f\right)(x) = g^{(1)}(x), \forall x \in A.$ □

Example: The Schwartz space $S(\mathbb{R}^d)$.
The *Schwartz space* or *space of rapidly decreasing functions* on $\mathbb{R}^d$ is defined as the set $\mathcal{S}(\mathbb{R}^d)$ of all real-valued functions which are defined and infinitely differentiable on $\mathbb{R}^d$ and which have the additional property (regulating their growth at infinity) that all their derivatives tend to zero at infinity faster than any inverse power of x, i.e.

$$\mathcal{S}(\mathbb{R}^d) := \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) : \sup_{x\in\mathbb{R}^d} \left| x^\alpha (D^\beta f)(x) \right| < \infty,\ \forall \alpha, \beta \in \mathbb{N}_0^d \right\}.$$

(For example, any smooth function f with compact support in $\mathbb{R}^d$ is in $\mathcal{S}(\mathbb{R}^d)$, since any derivative of f is continuous and supported on a compact subset of $\mathbb{R}^d$, so $x^\alpha(D^\beta f(x))$ has a maximum in $\mathbb{R}^d$ by the extreme value theorem.)

The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is a vector space over $\mathbb{R}$ and we equip it with the topology $\tau_{\mathcal{Q}}$ given by the family $\mathcal{Q}$ of seminorms on $\mathcal{S}(\mathbb{R}^d)$:

$$q_{m,k}(f) := \sup_{\substack{\beta\in\mathbb{N}_0^d\\ |\beta|\leq m}} \sup_{x\in\mathbb{R}^d} (1+|x|)^k \left|(D^\beta)f(x)\right|, \ \forall m,k\in\mathbb{N}_0.$$

Note that $f\in\mathcal{S}(\mathbb{R}^d)$ if and only if $\forall m,k\in\mathbb{N}_0, q_{m,k}(f)<\infty$.
The space $\mathcal{S}(\mathbb{R}^d)$ is a linear subspace of $\mathcal{C}^\infty(\mathbb{R}^d)$, but $\tau_{\mathcal{Q}}$ is finer than the subspace topology induced on it by $\tau_{\mathcal{P}}$ where $\mathcal{P}$ is the family of seminorms defined on $\mathcal{C}^\infty(\mathbb{R}^d)$ as in the above example. Indeed, it is clear that for any $f\in\mathcal{S}(\mathbb{R}^d)$, any $m\in\mathbb{N}_0$ and any $K\subset\mathbb{R}^d$ compact we have $p_{m,K}(f)\leq q_{m,0}(f)$ which gives the desired inclusion of topologies.

1) $(\mathcal{S}(\mathbb{R}^d),\tau_{\mathcal{Q}})$ is a locally convex t.v.s. which is also evidently Hausdorff since the family $\mathcal{Q}$ is separating. Indeed, if $q_{m,k}(f)=0$, $\forall m,k\in\mathbb{N}_0$ then in particular $q_{0,0}(f)=\sup_{x\in\mathbb{R}^d}|f(x)|=0$, which implies $f\equiv 0$ on $\mathbb{R}^d$.

2) $(\mathcal{S}(\mathbb{R}^d),\tau_{\mathcal{Q}})$ is a metrizable, as $\mathcal{Q}$ is countable and separating (see Proposition 1.1.5).

3) $(\mathcal{S}(\mathbb{R}^d),\tau_{\mathcal{Q}})$ is a complete. By Proposition 1.1.6, it is enough to show that it is sequentially complete. Let $(f_\nu)_{\nu\in\mathbb{N}}$ be a Cauchy sequence $\mathcal{S}(\mathbb{R}^d)$ then a fortiori we get that $(f_\nu)_{\nu\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}^\infty(\mathbb{R}^d)$ endowed with the $\mathcal{C}^\infty-$topology. Since such a space is complete, then there exists $f\in\mathcal{C}^\infty(\mathbb{R}^d)$ s.t. $(f_\nu)_{\nu\in\mathbb{N}}$ converges to f in the the $\mathcal{C}^\infty-$topology. From this we also know that:

$$\forall\beta\in\mathbb{N}_0^d, \forall x\in\mathbb{R}^d, (D^\beta f_\nu)(x)\to(D^\beta f)(x) \text{ as } \nu\to\infty \tag{1.11}$$

We are going to prove at once that $(f_\nu)_{\nu\in\mathbb{N}}$ is converging to f in the $\tau_{\mathcal{Q}}$ topology (not only in the $\mathcal{C}^\infty-$topology) and that $f\in\mathcal{S}(\mathbb{R}^d)$.

Let $m,k\in\mathbb{N}_0$ and let $\varepsilon>0$. As $(f_\nu)_{\nu\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}(\mathbb{R}^d)$, there exists a constant M s.t. $\forall\nu,\mu\geq M$ we have: $q_{m,k}(f_\nu-f_\mu)\leq\varepsilon$. Then fixing $\beta\in\mathbb{N}_0^d$ with $|\beta|\leq m$ and $x\in\mathbb{R}^d$ we get

$$(1+|x|)^k\left|(D^\beta f_\nu)(x)-(D^\beta f_\mu)(x)\right|\leq\varepsilon.$$

Passing to the limit for $\mu\to\infty$ in the latter relation and using (1.11), we get

$$(1+|x|)^k\left|(D^\beta f_\nu)(x)-(D^\beta f)(x)\right|\leq\varepsilon.$$

Hence, for all $\nu\geq M$ we have that $q_{m,k}(f_\nu-f)\leq\varepsilon$ as desired. Then by the triangular inequality it easily follows that

$$\forall m,k\in\mathbb{N}_0, q_{m,k}(f)<\infty, \text{ i.e. } f\in\mathcal{S}(\mathbb{R}^d).$$

1.3 Inductive topologies and LF-spaces

Let $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ be a family of locally convex t.v.s. over the field $\mathbb{K}$ of real or complex numbers (A is an arbitrary index set), E a vector space over the same field $\mathbb{K}$ and, for each $\alpha \in A$, let $g_\alpha : E_\alpha \to E$ be a linear mapping. The ***inductive topology*** τ_{ind} on E w.r.t. the family $\{(E_\alpha, \tau_\alpha, g_\alpha) : \alpha \in A\}$ is the locally convex topology generated by the following basis of neighbourhoods of the origin in E:

$$\begin{aligned}\mathcal{B}_{ind} : \quad = \quad & \{U \subset E \,\text{convex, balanced, absorbing} : \forall \alpha \in A,\ g_\alpha^{-1}(U) \ \text{ is} \\ & \text{a neighbourhood of the origin in}\, (E_\alpha, \tau_\alpha)\}.\end{aligned}$$

Hence, the space (E, τ_{ind}) is a l.c. t.v.s.. Indeed, $\mathcal{B}_{ind}$ is a family of absorbing and absolutely convex subsets of E such that

a) $\forall\, U, V \in \mathcal{B}_{ind}$, $U \cap V \in \mathcal{B}_{ind}$, since $g_\alpha^{-1}(U \cap V) = g_\alpha^{-1}(U) \cap g_\alpha^{-1}(V)$ is a neighbourhood of the origin in (E_α, τ_α) (as finite intersection of such neighbourhoods).

b) $\forall\, \rho > 0, \forall\, U \in \mathcal{B}_{ind}$, $\rho U \in \mathcal{B}_{ind}$, since $g_\alpha^{-1}(\rho U) = \rho g_\alpha^{-1}(U)$ which is a neighbourhood of the origin in (E_α, τ_α)(as a dilation of such a neighbourhood).

Then Theorem 4.1.14 in TVS-I ensures that τ_{ind} makes E into a l.c. t.v.s..

Note that τ_{ind} is the finest locally convex topology on E for which all the mappings g_α ($\alpha \in A$) are continuous. Suppose there exists a locally convex topology τ on E s.t. all the g_α's are continuous and $\tau_{ind} \subseteq \tau$. As (E, τ) is locally convex, there always exists a basis of neighbourhood of the origin consisting of convex, balanced, absorbing subsets of E. Then for any such a neighbourhood U of the origin in (E, τ) we have, by continuity, that $g_\alpha^{-1}(U)$ is a neighbourhood of the origin in (E_α, τ_α). Hence, $U \in \mathcal{B}_{ind}$ and so $\tau \equiv \tau_{ind}$.

It is also worth to underline that (E, τ_{ind}) is not necessarily a Hausdorff t.v.s., even when all the spaces (E_α, τ_α) are Hausdorff t.v.s..

Example 1.3.1. *Let (X, τ) be a l.c. Hausdorff t.v.s., M a non-closed subspace of X and $\varphi : X \to X/M$ the quotient map. Then the inductive limit topology on X/M w.r.t. (X, τ, ϕ) (here the index set A is just a singleton) coincides with the quotient topology on X/M, which is not Hausdorff since M is not closed (see Proposition 2.3.5 in TVS-I).*

Proposition 1.3.2. *Let E be a vector space over the field $\mathbb{K}$ endowed with the inductive topology τ_{ind} w.r.t. a family $\{(E_\alpha, \tau_\alpha, g_\alpha) : \alpha \in A\}$, where each (E_α, τ_α) is a locally convex t.v.s. over $\mathbb{K}$ and each $g_\alpha : E_\alpha \to E$ is a linear map. A linear map u from (E, τ_{ind}) to any locally convex t.v.s. (F, τ) is continuous if and only if for each $\alpha \in A$ the map $u \circ g_\alpha : E_\alpha \to F$ is continuous.*

Proof. Suppose u is continuous and fix $\alpha \in A$. Since g_α is also continuous, we have that $u \circ g_\alpha$ is continuous as composition of continuous mappings.

Conversely, suppose that for each $\alpha \in A$ the mapping $u \circ g_\alpha$ is continuous. As (F, τ) is locally convex, there always exists a basis of neighbourhoods of the origin consisting of convex, balanced, absorbing subsets of F. Let W be such a neighbourhood. Then, by the linearity of u, we get that $u^{-1}(W)$ is a convex, balanced and absorbing subset of E. Moreover, the continuity of all $u \circ g_\alpha$ guarantees that each $(u \circ g_\alpha)^{-1}(W)$ is a neighbourhood of the origin in (E_α, τ_α), i.e. $g_\alpha^{-1}(u^{-1}(W))$ is a neighbourhood of the origin in (E_α, τ_α). Then $u^{-1}(W)$, being also convex, balanced and absorbing, must be in $\mathcal{B}_{ind}$ and so it is a neighbourhood of the origin in (E, τ_{ind}). Hence, u is continuous. $\square$

Let us consider now the case when we have a total order $\leq$ on the index set A and $\{E_\alpha : \alpha \in A\}$ is a family of linear subspaces of a vector space E over $\mathbb{K}$ which is directed under inclusion, i.e. $E_\alpha \subseteq E_\beta$ whenever $\alpha \leq \beta$, and s.t. $E = \bigcup_{\alpha \in A} E_\alpha$. For each $\alpha \in A$, let i_α be the canonical embedding of E_α in E and τ_α a topology on E_α s.t. (E_α, τ_α) is a locally convex Hausdorff t.v.s. and, whenever $\alpha \leq \beta$, the topology induced by τ_β on E_α is coarser than τ_α. The space E equipped with the inductive topology τ_{ind} w.r.t. the family $\{(E_\alpha, \tau_\alpha, i_\alpha) : \alpha \in A\}$ is said to be the ***inductive limit*** of the family of linear subspaces $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$.

An inductive limit of a family of linear subspaces $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ is said to be a ***strict inductive limit*** if, whenever $\alpha \leq \beta$, the topology induced by τ_β on E_α coincides with τ_α.

There are even more general constructions of inductive limits of a family of locally convex t.v.s. but in the following we will focus on a more concrete family of inductive limits which are more common in applications. Namely, we are going to consider the so-called ***LF-spaces***, i.e. countable strict inductive limits of increasing sequences of Fréchet spaces. For convenience, let us explicitly write down the definition of an LF-space.

Definition 1.3.3. *Let $\{E_n : n \in \mathbb{N}\}$ be an increasing sequence of linear subspaces of a vector space E over $\mathbb{K}$, i.e. $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$, such that $E = \bigcup_{n \in \mathbb{N}} E_n$. For each $n \in \mathbb{N}$ let i_n be the canonical embedding of E_n in E and (E_n, τ_n) be a Fréchet space such that the topology induced by τ_{n+1} on E_n coincides with τ_n (i.e. the natural embedding of E_n into E_{n+1} is a topological embedding). The space E equipped with the inductive topology τ_{ind} w.r.t. the family $\{(E_n, \tau_n, i_n) : n \in \mathbb{N}\}$ is said to be the* LF-space with defining sequence *$\{(E_n, \tau_n) : n \in \mathbb{N}\}$.*

A basis of neighbourhoods of the origin in the LF-space (E, τ_{ind}) with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$ is given by:

$\{U \subset E$ convex, balanced, abs. : $\forall n \in \mathbb{N}$, $U \cap E_n$ is a nbhood of o in $(E_n, \tau_n)\}$.

Note that from the construction of the LF-space (E, τ_{ind}) with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$ we know that each E_n is topologically embedded in the subsequent ones, but a priori we do not know if E_n is topologically embedded in E, i.e. if the topology induced by τ_{ind} on E_n is identical to the topology τ_n initially given on E_n. This is indeed true and it will be a consequence of the following lemma.

Lemma 1.3.4. *Let X be a locally convex t.v.s., X_0 a linear subspace of X equipped with the subspace topology, and U a convex neighbourhood of the origin in X_0. Then there exists a convex neighbourhood V of the origin in X such that $V \cap X_0 = U$.*

Proof.
As X_0 carries the subspace topology induced by X, there exists a neighbourhood W of the origin in X such that $U = W \cap X_0$. Since X is a locally convex t.v.s., there exists a convex neighbourhood W_0 of the origin in X such that $W_0 \subseteq W$. Let V be the convex hull of $U \cup W_0$. Then by construction we have that V is a convex neighbourhood of the origin in X and that $U \subseteq V$ which implies $U = U \cap X_0 \subseteq V \cap X_0$. We claim that actually $V \cap X_0 = U$. Indeed, let $x \in V \cap X_0$; as $x \in V$ and as U and W_0 are both convex, we may write $x = ty + (1-t)z$ with $y \in U, z \in W_0$ and $t \in [0,1]$. If $t = 1$, then $x = y \in U$ and we are done. If $0 \leq t < 1$, then $z = (1-t)^{-1}(x - ty)$ belongs to X_0 and so $z \in W_0 \cap X_0 \subseteq W \cap X_0 = U$. This implies, by the convexity of U, that $x \in U$. Hence, $V \cap X_0 \subseteq U$. ☐

Proposition 1.3.5.
Let (E, τ_{ind}) be an LF-space with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$. Then

$$\tau_{ind} \restriction E_n \equiv \tau_n, \forall n \in \mathbb{N}.$$

Proof.
($\subseteq$) Let $V \in \mathcal{B}_{ind}$. Then, by definition, for each $n \in \mathbb{N}$ we have that $V \cap E_n$ is a neighbourhood of the origin in (E_n, τ_n). Hence, $\tau_{ind} \restriction E_n \subseteq \tau_n$, $\forall n \in \mathbb{N}$.

($\supseteq$) Given $n \in \mathbb{N}$, let U_n be a convex, balanced, absorbing neighbourhood of the origin in (E_n, τ_n). Since E_n is a linear subspace of E_{n+1}, we can apply Lemma 1.3.4 (for $X = E_{n+1}$, $X_0 = E_n$ and $U = U_n$) which ensures the existence of a convex neighbourhood U_{n+1} of the origin in (E_{n+1}, τ_{n+1}) such that $U_{n+1} \cap E_n = U_n$. Then, by induction, we get that for any $k \in \mathbb{N}$ there exists a convex neighbourhood U_{n+k} of the origin in (E_{n+k}, τ_{n+k}) such that $U_{n+k} \cap E_{n+k-1} = U_{n+k-1}$. Hence, for any $k \in \mathbb{N}$, we get $U_{n+k} \cap E_n = U_n$. If we consider now $U := \bigcup_{k=1}^{\infty} U_{n+k}$, then $U \cap E_n = U_n$ and U is a neighbourhood of the origin in (E, τ_{ind}). Indeed, for any $m \in \mathbb{N}$ we have $U \cap E_m = \bigcup_{k=1}^{\infty} U_{n+k} \cap E_m = \bigcup_{k=m-n}^{\infty} U_{n+k} \cap E_m$, which is a countable union of neighbourhoods of the origin in τ_m as for $k \geq m-n$ we get $n+k \geq m$ and so $\tau_{n+k} \restriction E_m = \tau_m$. We can then conclude that $\tau_n \subseteq \tau_{ind} \restriction E_n$, $\forall n \in \mathbb{N}$. □

Corollary 1.3.6. *Any LF-space is a locally convex Hausdorff. t.v.s..*

Proof. Let (E, τ_{ind}) be the LF-space with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$ and denote by $\mathcal{F}(o)$ [resp. $\mathcal{F}_n(o)$] the filter of neighbourhoods of the origin in (E, τ_{ind}) [resp. in (E_n, τ_n)]. Then:

$$\bigcap_{V \in \mathcal{F}(o)} V = \bigcap_{V \in \mathcal{F}(o)} V \cap \left(\bigcup_{n \in \mathbb{N}} E_n \right) = \bigcup_{n \in \mathbb{N}} \bigcap_{V \in \mathcal{F}(o)} (V \cap E_n) = \bigcup_{n \in \mathbb{N}} \bigcap_{U_n \in \mathcal{F}_n(o)} U_n = \{o\},$$

which implies that (E, τ_{ind}) is Hausdorff by Corollary 2.2.4 in TVS-I. □

As a particular case of Proposition 1.3.2 we easily get that:

Proposition 1.3.7.
Let (E, τ_{ind}) be an LF-space with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$ and (F, τ) an arbitrary locally convex t.v.s..

1. *A linear mapping u from E into F is continuous if and only if, for each $n \in \mathbb{N}$, the restriction $u \restriction E_n$ of u to E_n is continuous.*
2. *A linear form on E is continuous if and only if its restrictions to each E_n are continuous.*

Note that Propositions 1.3.5 and 1.3.7 and Corollary 1.3.6 hold for any countable strict inductive limit of an increasing sequence of locally convex Hausdorff t.v.s. (even when they are not Fréchet).

The next theorem is instead typical of LF-spaces as it heavily relies on the completeness of the t.v.s. of the defining sequence. Before introducing it, let us recall the concept of accumulation point of a filter on a topological space together with some basic useful properties.

Definition 1.3.8. *Let $\mathcal{F}$ be a filter on a topological space X. A point $x \in X$ is called an* accumulation point *of $\mathcal{F}$ if x belongs to the closure of every set which belongs to $\mathcal{F}$, i.e. $x \in \overline{M}$, $\forall M \in \mathcal{F}$.*

Proposition 1.3.9. *If a filter $\mathcal{F}$ of a topological space X converges to a point x, then x is an accumulation point of $\mathcal{F}$.*

Proof. If x were not an accumulation point of $\mathcal{F}$, then there would be a set $M \in \mathcal{F}$ such that $x \notin \overline{M}$. Hence, $X \setminus \overline{M}$ is open in X and contains x, so it is a neighbourhood of x. Then $X \setminus \overline{M} \in \mathcal{F}$ as $\mathcal{F} \to x$ by assumption. But $\mathcal{F}$ is a filter and so $M \cap \left(X \setminus \overline{M}\right) \in \mathcal{F}$ and so $M \cap \left(X \setminus \overline{M}\right) \neq \emptyset$, which is a contradiction. □

Proposition 1.3.10. *If a Cauchy filter $\mathcal{F}$ of a t.v.s. X has an accumulation point x, then $\mathcal{F}$ converges to x.*

Proof. Let us denote by $\mathcal{F}(o)$ the filter of neighbourhoods of the origin in X and consider $U \in \mathcal{F}(o)$. Since X is a t.v.s., there exists $V \in \mathcal{F}(o)$ such that $V + V \subseteq U$. Then there exists $M \in \mathcal{F}$ such that $M - M \subseteq V$ as $\mathcal{F}$ is a Cauchy filter in X. Being x an accumulation point of $\mathcal{F}$ guarantees that $x \in \overline{M}$ and so that $(x + V) \cap M \neq \emptyset$. Then $M - ((x + V) \cap M) \subseteq M - M \subseteq V$ and so $M \subseteq V + ((x + V) \cap M) \subseteq V + V + x \subseteq U + x$. Since $\mathcal{F}$ is a filter and $M \in \mathcal{F}$, the latter implies that $U + x \in \mathcal{F}$. This proves that $\mathcal{F}(x) \subseteq \mathcal{F}$, i.e. $\mathcal{F} \to x$. □

Theorem 1.3.11. *Any LF-space is complete.*

Proof.
Let (E, τ_{ind}) be an LF-space with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$. Let $\mathcal{F}$ be a Cauchy filter on (E, τ_{ind}). Denote by $\mathcal{F}_E(o)$ the filter of neighbourhoods of the origin in (E, τ_{ind}) and consider

$$\mathcal{G} := \{A \subseteq E : \; A \supseteq M + V \; \text{ for some } M \in \mathcal{F}, V \in \mathcal{F}_E(o)\}.$$

1) $\mathcal{G}$ is a filter on E.
Indeed, it is clear from its definition that $\mathcal{G}$ does not contain the empty set and that any subset of E containing a set in $\mathcal{G}$ has to belong to $\mathcal{G}$. Moreover, for any $A_1, A_2 \in \mathcal{G}$ there exist $M_1, M_2 \in \mathcal{F}$, $V_1, V_2 \in \mathcal{F}_E(o)$ s.t. $M_1 + V_1 \subseteq A_1$ and $M_2 + V_2 \subseteq A_2$; and therefore

$$A_1 \cap A_2 \supseteq (M_1 + V_1) \cap (M_2 + V_2) \supseteq (M_1 \cap M_2) + (V_1 \cap V_2).$$

The latter proves that $A_1 \cap A_2 \in \mathcal{G}$, since $\mathcal{F}$ and $\mathcal{F}_E(o)$ are both filters and so $M_1 \cap M_2 \in \mathcal{F}$ and $V_1 \cap V_2 \in \mathcal{F}_E(o)$.

2) $\mathcal{G} \subseteq \mathcal{F}$.
In fact, for any $A \in \mathcal{G}$ there exist $M \in \mathcal{F}$ and $V \in \mathcal{F}_E(o)$ s.t.

$$A \supseteq M + V \supset M + \{0\} = M$$

which implies that $A \in \mathcal{F}$ since $\mathcal{F}$ is a filter.

3) $\mathcal{G}$ is a Cauchy filter on E.
Let $U \in \mathcal{F}_E(o)$. Then there always exists $V \in \mathcal{F}_E(o)$ balanced such that $V + V - V \subseteq U$. As $\mathcal{F}$ is a Cauchy filter on (E, τ_{ind}), there exists $M \in \mathcal{F}$ such that $M - M \subseteq V$. Then

$$(M + V) - (M + V) \subseteq (M - M) + (V - V) \subseteq V + V - V \subseteq U$$

which proves that $\mathcal{G}$ is a Cauchy filter since $M + V \in \mathcal{G}$.

It is possible to show (and we do it later on) that

$$\exists\, p \in \mathbb{N} :\ \forall A \in \mathcal{G},\ A \cap E_p \neq \emptyset. \tag{1.12}$$

This property together with the fact that $\mathcal{G}$ is a filter ensures that the family

$$\mathcal{G}_p := \{A \cap E_p :\ A \in \mathcal{G}\}$$

is a filter on E_p. Moreover, since $\mathcal{G}$ is a Cauchy filter on (E, τ_{ind}) and since by Proposition 1.3.5 we have $\tau_{ind} \restriction E_p = \tau_p$, $\mathcal{G}_p$ is a Cauchy filter on (E_p, τ_p). Hence, the completeness of E_p guarantees that there exists $x \in E_p$ s.t. $\mathcal{G}_p \to x$ which implies in turn that x is an accumulation point for $\mathcal{G}_p$ by Proposition 1.3.9. In particular, this gives that for any $A \in \mathcal{G}$ we have $x \in \overline{A \cap E_p}^{\tau_p} = \overline{A \cap E_p}^{\tau_{ind}} \subseteq \overline{A}^{\tau_{ind}}$, i.e. x is an accumulation point for the Cauchy filter $\mathcal{G}$. Then, by Proposition 1.3.10, we get that $\mathcal{G} \to x$ and so $\mathcal{F}_E(x) \subseteq \mathcal{G} \subseteq \mathcal{F}$. Hence, we proved that $\mathcal{F} \to x \in E$. □

Proof. of (1.12)
Suppose that (1.12) is false, i.e. $\forall n \in \mathbb{N}$, $\exists\, A_n \in \mathcal{G}$ s.t. $A_n \cap E_n = \emptyset$. By the definition of $\mathcal{G}$, this implies that

$$\forall n \in \mathbb{N},\ \exists\, M_n \in \mathcal{F},\ V_n \in \mathcal{F}_E(o), \text{ s.t. } (M_n + V_n) \cap E_n = \emptyset. \tag{1.13}$$

Since E is a locally convex t.v.s., we may assume that each V_n is balanced, convex, and such that $V_{n+1} \subseteq V_n$. For each $n \in \mathbb{N}$, define

$$W_n := conv\left(V_n \cup \bigcup_{k=1}^{n-1} (V_k \cap E_k)\right).$$

Moreover, if for some $n \in \mathbb{N}$ there exists $h \in (W_n + M_n) \cap E_n$ then $h \in E_n$ and $h \in (W_n + M_n)$. Therefore, we can write $h = x + w$ with $x \in M_n$ and $w \in W_n \subseteq conv\,(V_n \cup (V_1 \cap E_{n-1}))$. As V_n and $V_1 \cap E_{n-1}$ are both convex, we get that $h = x + ty + (1-t)z$ with $x \in M_n$, $y \in V_n$, $z \in V_1 \cap E_{n-1}$ and $t \in [0,1]$. Then $x + ty = h - (1-t)z \in E_n$, but we also have $x + ty \in M_n + V_n$ (since V_n is balanced). Hence, $x + ty \in (M_n + V_n) \cap E_n$ which contradicts (1.13), proving that

$$(W_n + M_n) \cap E_n = \emptyset, \forall n \in \mathbb{N}.$$

Now let us define

$$W := conv\left(\bigcup_{k=1}^{\infty}(V_k \cap E_k)\right).$$

As W is convex and as $W \cap E_k$ contains $V_k \cap E_k$ for all $k \in \mathbb{N}$, W is a neighbourhood of the origin in (E, τ_{ind}). Moreover, as $(V_n)_{n\in\mathbb{N}}$ is decreasing, we have that for all $n \in \mathbb{N}$

$$W = conv\left(\bigcup_{k=1}^{n-1}(V_k \cap E_k) \cup \bigcup_{k=n}^{\infty}(V_k \cap E_k)\right) \subseteq conv\left(\bigcup_{k=1}^{n-1}(V_k \cap E_k) \cup V_n\right) = W_n.$$

Since $\mathcal{F}$ is a Cauchy filter on (E, τ_{ind}), there exists $B \in \mathcal{F}$ such that $B - B \subseteq W$ and so $B - B \subseteq W_n, \forall n \in \mathbb{N}$. We also have that $B \cap M_n \neq \emptyset, \forall n \in \mathbb{N}$, as both B and M_n belong to $\mathcal{F}$. Hence, for all $n \in \mathbb{N}$ we get

$$B - (B \cap M_n) \subseteq B - B \subseteq W_n,$$

which implies

$$B \subseteq W_n + (B \cap M_n) \subseteq W_n + M_n$$

and so

$$B \cap E_n \subseteq (W_n + M_n) \cap E_n \overset{(1.13)}{=} \emptyset.$$

Therefore, we have got that $B \cap E_n = \emptyset$ for all $n \in \mathbb{N}$ and so that $B = \emptyset$, which is impossible as $B \in \mathcal{F}$. Hence, (1.12) must hold true. □

Example I: The space of polynomials

Let $n \in \mathbb{N}$ and $\mathbf{x} := (x_1, \ldots, x_n)$. Denote by $\mathbb{R}[\mathbf{x}]$ the space of polynomials in the n variables $x_1, \ldots, x_n$ with real coefficients. A canonical algebraic basis for $\mathbb{R}[\mathbf{x}]$ is given by all the monomials

$$\mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \forall \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n.$$

For any $d \in \mathbb{N}_0$, let $\mathbb{R}_d[\mathbf{x}]$ be the linear subpace of $\mathbb{R}[\mathbf{x}]$ spanned by all monomials $\mathbf{x}^\alpha$ with $|\alpha| := \sum_{i=1}^n \alpha_i \leq d$, i.e.

$$\mathbb{R}_d[\mathbf{x}] := \{f \in \mathbb{R}[\mathbf{x}] |\deg f \leq d\}.$$

Since there are exactly $\binom{n+d}{d}$ monomials $\mathbf{x}^\alpha$ with $|\alpha| \leq d$, we have that

$$dim(\mathbb{R}_d[\mathbf{x}]) = \frac{(d+n)!}{d!n!},$$

and so that $\mathbb{R}_d[\mathbf{x}]$ is a finite dimensional vector space. Hence, by Tychonoff Theorem (see Corollary 3.1.4 in TVS-I) there is a unique topology τ_e^d that makes $\mathbb{R}_d[\mathbf{x}]$ into a Hausdorff t.v.s. which is also complete and so Fréchet (as it topologically isomorphic to $\mathbb{R}^{dim(\mathbb{R}_d[\underline{x}])}$ equipped with the euclidean topology).

As $\mathbb{R}[\mathbf{x}] := \bigcup_{d=0}^\infty \mathbb{R}_d[\mathbf{x}]$, we can then endow it with the inductive topology τ_{ind} w.r.t. the family of F-spaces $\left\{(\mathbb{R}_d[\mathbf{x}], \tau_e^d) : d \in \mathbb{N}_0\right\}$; thus $(\mathbb{R}[\mathbf{x}], \tau_{ind})$ is a LF-space and the following properties hold (proof in Exercise Sheet 3):

a) τ_{ind} is the finest locally convex topology on $\mathbb{R}[\mathbf{x}]$,

b) every linear map f from $(\mathbb{R}[\mathbf{x}], \tau_{ind})$ into any t.v.s. is continuous.

Example II: The space of test functions

Let $\Omega \subseteq \mathbb{R}^d$ be open in the euclidean topology. For any integer $0 \leq s \leq \infty$, we have defined in Section 1.2 the set $\mathcal{C}^s(\Omega)$ of all real valued *$s-$times continuously differentiable functions* on Ω, which is a real vector space w.r.t. pointwise addition and scalar multiplication. We have equipped this space with the *$\mathcal{C}^s$-topology* (i.e. the topology of uniform convergence on compact sets of the functions and their derivatives up to order s) and showed that this turns $\mathcal{C}^s(\Omega)$ into a Fréchet space.

Let K be a compact subset of Ω, which means that it is bounded and closed in $\mathbb{R}^d$ and that its closure is contained in Ω. For any integer $0 \leq s \leq \infty$, consider the subset $\mathcal{C}_c^k(K)$ of $\mathcal{C}^s(\Omega)$ consisting of all the functions $f \in \mathcal{C}^s(\Omega)$ whose support lies in K, i.e.

$$\mathcal{C}_c^s(K) := \{f \in \mathcal{C}^s(\Omega) : supp(f) \subseteq K\},$$

where $supp(f)$ denotes the support of the function f on Ω, that is the closure in Ω of the subset $\{x \in \Omega : f(x) \neq 0\}$.

For any integer $0 \leq s \leq \infty$, $\mathcal{C}_c^s(K)$ is always a closed linear subspace of $\mathcal{C}^s(\Omega)$. Indeed, for any $f, g \in \mathcal{C}_c^s(K)$ and any $\lambda \in \mathbb{R}$, we clearly have $f + g \in \mathcal{C}^s(\Omega)$ and $\lambda f \in \mathcal{C}^s(\Omega)$ but also $supp(f + g) \subseteq supp(f) \cup supp(g) \subseteq K$ and $supp(\lambda f) = supp(f) \subseteq K$, which gives $f + g, \lambda f \in \mathcal{C}_c^s(K)$. To show that $\mathcal{C}_c^s(K)$ is closed in $\mathcal{C}^s(\Omega)$, it suffices to prove that it is sequentially closed as $\mathcal{C}^s(\Omega)$ is a F-space. Consider a sequence $(f_j)_{j\in\mathbb{N}}$ of functions in $\mathcal{C}^s_c(K)$ converging to f in the $\mathcal{C}^s-$topology. Then clearly $f \in \mathcal{C}^s(\Omega)$ and since all the f_j vanish in the open set $\Omega \setminus K$, obviously their limit f must also vanish in $\Omega \setminus K$. Thus, regarded as a subspace of $\mathcal{C}^s(\Omega)$, $\mathcal{C}^s_c(K)$ is also complete (see Proposition 2.5.8 in TVS-I) and so it is itself an F-space.

Let us now denote by $\mathcal{C}^s_c(\Omega)$ the union of the subspaces $\mathcal{C}^s_c(K)$ as K varies in all possible ways over the family of compact subsets of Ω, i.e. $\mathcal{C}^s_c(\Omega)$ is linear subspace of $\mathcal{C}^s(\Omega)$ consisting of all the functions belonging to $\mathcal{C}^s(\Omega)$ which have a compact support (this is what is actually encoded in the subscript c). In particular, $\mathcal{C}^\infty_c(\Omega)$ (smooth functions with compact support in Ω) is called *space of test functions* and plays an essential role in the theory of distributions.

We will not endow $\mathcal{C}^s_c(\Omega)$ with the subspace topology induced by $\mathcal{C}^s(\Omega)$, but we will consider a finer one, which will turn $\mathcal{C}^s_c(\Omega)$ into an LF-space. Let us consider a sequence $(K_j)_{j\in\mathbb{N}}$ of compact subsets of Ω s.t. $K_j \subseteq K_{j+1}, \forall j \in \mathbb{N}$ and $\bigcup_{j=1}^\infty K_j = \Omega$. (Sometimes is even more advantageous to choose the K_j's to be relatively compact i.e. the closures of open subsets of Ω such that $K_j \subseteq \mathring{K}_{j+1}, \forall j \in \mathbb{N}$ and $\bigcup_{j=1}^\infty K_j = \Omega$.)

Then $\mathcal{C}^s_c(\Omega) = \bigcup_{j=1}^\infty \mathcal{C}^s_c(K_j)$, as an arbitrary compact subset K of Ω is contained in K_j for some sufficiently large j. Because of our way of defining the F-spaces $\mathcal{C}^s_c(K_j)$, we have that $\mathcal{C}^s_c(K_j) \subseteq \mathcal{C}^s_c(K_{j+1})$ and $\mathcal{C}^s_c(K_{j+1})$ induces on the subset $\mathcal{C}^s_c(K_j)$ the same topology as the one originally given on it, i.e. the subspace topology induced on $\mathcal{C}^s_c(K_j)$ by $\mathcal{C}^s(\Omega)$. Thus we can equip $\mathcal{C}^s_c(\Omega)$ with the inductive topology τ_{ind} w.r.t. the sequence of F-spaces $\{\mathcal{C}^s_c(K_j),\, j \in \mathbb{N}\}$, which makes $\mathcal{C}^s_c(\Omega)$ an LF-space. It is easy to check that τ_{ind} does not depend on the choice of the sequence of compact sets K_j's provided they fill Ω.

Note that $(\mathcal{C}^s_c(\Omega), \tau_{ind})$ is not metrizable since it is not Baire (proof in Exercise Sheet 3).

Proposition 1.3.12. *For any integer $0 \leq s \leq \infty$, consider $\mathcal{C}^s_c(\Omega)$ endowed with the LF-topology τ_{ind} described above. Then we have the following continuous injections:*

$$\mathcal{C}^\infty_c(\Omega) \to \mathcal{C}^s_c(\Omega) \to \mathcal{C}^{s-1}_c(\Omega), \quad \forall\, 0 < s < \infty.$$

Proof. Let us just prove the first inclusion $i : \mathcal{C}^\infty_c(\Omega) \to \mathcal{C}^s_c(\Omega)$ as the others follows in the same way. As $\mathcal{C}^\infty_c(\Omega) = \bigcup_{j=1}^\infty \mathcal{C}^\infty_c(K_j)$ is the inductive limit of the sequence of F-spaces $(\mathcal{C}^\infty_c(K_j))_{j\in\mathbb{N}}$, where $(K_j)_{j\in\mathbb{N}}$ is a sequence of compact subsets of Ω such that $K_j \subseteq K_{j+1}, \forall j \in \mathbb{N}$ and $\bigcup_{j=1}^\infty K_j = \Omega$, by Proposition 1.3.7 we know that i is continuous if and only if, for any $j \in \mathbb{N}$, $e_j := i \restriction \mathcal{C}^\infty_c(K_j)$ is continuous. But from the definition we gave of the topology on each $\mathcal{C}^s_c(K_j)$ and $\mathcal{C}^\infty_c(K_j)$, it is clear that both the inclusions $i_j : \mathcal{C}^\infty_c(K_j) \to \mathcal{C}^s_c(K_j)$ and $s_j : \mathcal{C}^s_c(K_j) \to \mathcal{C}^s_c(\Omega)$ are continuous. Hence, for each $j \in \mathbb{N}$, $e_j = s_j \circ i_j$ is indeed continuous. □

1.4 Projective topologies and examples of projective limits

Let $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ be a family of locally convex t.v.s. over the field $\mathbb{K}$ of real or complex numbers (A is an arbitrary index set). Let E be a vector space over the same field $\mathbb{K}$ and, for each $\alpha \in A$, let $f_\alpha : E \to E_\alpha$ be a linear mapping. The ***projective topology*** τ_{proj} on E w.r.t. the family $\{(E_\alpha, \tau_\alpha, f_\alpha) : \alpha \in A\}$ is the locally convex topology generated by the following basis of neighbourhoods of the origin in E:

$$\mathcal{B}_{proj} := \left\{ \bigcap_{\alpha\in F} f_\alpha^{-1}(U_\alpha) : \; F \subseteq A \,\text{finite}, \, U_\alpha \text{ basic nbhood of o in} \,(E_\alpha, \tau_\alpha), \, \forall \alpha \in F \right\}.$$

Hence, (E, τ_{proj}) is a locally convex t.v.s.. Indeed, since all (E_α, τ_α) are locally convex t.v.s., we can always choose the U_α's to be convex, balanced and absorbing and so, by the linearity of the f_α's, we get that the corresponding $\mathcal{B}_{proj}$ is a collection of convex, balanced and absorbing subsets of E such that:

a) $\forall\; U, V \in \mathcal{B}_{proj}$, $U \cap V \in \mathcal{B}_{proj}$, because $U = \bigcap_{\alpha\in F} f_\alpha^{-1}(U_\alpha)$ and $V = \bigcap_{\alpha\in G} f_\alpha^{-1}(U_\alpha)$ for some $F, G \subseteq A$ finite and some U_α basic neighbourhoods of the origin in (E_α, τ_α) and so $U \cap V = \bigcap_{\alpha\in F\cup G} f_\alpha^{-1}(U_\alpha) \in \mathcal{B}_{proj}$.

b) $\forall\, \rho > 0, \forall\, U \in \mathcal{B}_{proj}$, $\rho U \in \mathcal{B}_{proj}$, since $U = \bigcap_{\alpha\in F} f_\alpha^{-1}(U_\alpha)$ for some $F \subseteq A$ finite and some U_α basic neighbourhoods of the origin in (E_α, τ_α) and so $\rho U = \bigcap_{\alpha\in F} f_\alpha^{-1}(\rho U_\alpha) \in \mathcal{B}_{proj}$.

Then Theorem 4.1.14 in TVS-I ensures that τ_{proj} makes E into a l.c. t.v.s..

Note that τ_{proj} is the coarsest topology on E for which all the mappings f_α ($\alpha \in A$) are continuous. Suppose there exists another topology τ on E such that all the f_α's are continuous and $\tau \subseteq \tau_{proj}$. Then for any neighbourhood U of the origin in τ_{proj} there exists $F \subseteq A$ finite and for each $\alpha \in F$ there exists U_α basic neighbourhood of the origin in (E_α, τ_α) such that $\bigcap_{\alpha\in F} f_\alpha^{-1}(U_\alpha) \subseteq U$. Since the τ-continuity of the f_α's ensures that each $f_\alpha^{-1}(U_\alpha)$ is a neighbourhood of the origin in τ, we have that U is itself a neighbourhood of the origin in τ. Hence, $\tau \equiv \tau_{proj}$.

Proposition 1.4.1. *Let E be a vector space over $\mathbb{K}$ endowed with the projective topology τ_{proj} w.r.t. the family $\{(E_\alpha, \tau_\alpha, f_\alpha) : \alpha \in A\}$, where each (E_α, τ_α) is a locally convex t.v.s. over $\mathbb{K}$ and each f_α a linear mapping from E to E_α. Then τ_{proj} is Hausdorff if and only if for each $0 \neq x \in E$, there exists an $\alpha \in A$ and a neighbourhood U_α of the origin in (E_α, τ_α) such that $f_\alpha(x) \notin U_\alpha$.*

Proof. Suppose that (E, τ_{proj}) is Hausdorff and let $0 \neq x \in E$. By Proposition 2.2.3 in TVS-I, there exists a neighbourhood U of the origin in E not containing x. Then, by definition of τ_{proj} there exists a finite subset $F \subseteq A$

and, for any $\alpha \in F$, there exists U_α neighbourhood of the origin in (E_α, τ_α) s.t. $\bigcap_{\alpha\in F} f_\alpha^{-1}(U_\alpha) \subseteq U$. Hence, as $x \notin U$, there exists $\alpha \in F$ s.t. $x \notin f_\alpha^{-1}(U_\alpha)$ i.e. $f_\alpha(x) \notin U_\alpha$. Conversely, suppose that there exists an $\alpha \in A$ and a neighbourhood of the origin in (E_α, τ_α) such that $f_\alpha(x) \notin U_\alpha$. Then $x \notin f_\alpha^{-1}(U_\alpha)$, which implies by Proposition 2.2.3 in TVS-I that τ_{proj} is a Hausdorff topology, as $f_\alpha^{-1}(U_\alpha)$ is a neighbourhood of the origin in (E, τ_{proj}) not containing x. □

Proposition 1.4.2. *Let E be a vector space over $\mathbb{K}$ endowed with the projective topology τ_{proj} w.r.t. the family $\{(E_\alpha, \tau_\alpha, f_\alpha) : \alpha \in A\}$, where each (E_α, τ_α) is a locally convex t.v.s. over $\mathbb{K}$ and each f_α a linear mapping from E to E_α. Let (F, τ) be an arbitrary t.v.s. and u a linear mapping from F into E. The mapping $u : F \to E$ is continuous if and only if, for each $\alpha \in A$, $f_\alpha \circ u : F \to E_\alpha$ is continuous.*

Proof. (Exercise Sheet 3) □

Example I: The product of locally convex t.v.s

Let $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ be a family of locally convex t.v.s. The product topology τ_{prod} on $E = \prod_{\alpha\in A} E_\alpha$ (see Definition 1.1.20 in TVS-I) is the coarsest topology for which all the canonical projections $p_\alpha : E \to E_\alpha$ (defined by $p_\alpha(x) := x_\alpha$ for any $x = (x_\beta)_{\beta\in A} \in E$) are continuous. Hence, τ_{prod} coincides with the projective topology on E w.r.t. $\{(E_\alpha, \tau_\alpha, p_\alpha) : \alpha \in A\}$.

Let us consider now the case when we have a directed partially ordered index set $(A, \leq)$, a family $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ of locally convex t.v.s. over $\mathbb{K}$ and for any $\alpha \leq \beta$ a continuous linear mapping $g_{\alpha\beta} : E_\beta \to E_\alpha$. Let E be the subspace of $\prod_{\alpha\in A} E_\alpha$ whose elements $x = (x_\alpha)_{\alpha\in A}$ satisfy the relation $x_\alpha = g_{\alpha\beta}(x_\beta)$ whenever $\alpha \leq \beta$. For any $\alpha \in A$, let f_α be the canonical projection $p_\alpha : \prod_{\beta\in A} E_\beta \to E_\alpha$ restricted to E. The space E endowed with the projective topology w.r.t. the family $\{(E_\alpha, \tau_\alpha, f_\alpha) : \alpha \in A\}$ is said to be the ***projective limit*** of the family $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ w.r.t. the mappings $\{g_{\alpha\beta} : \alpha, \beta \in A, \alpha \leq \beta\}$ and $\{f_\alpha : \alpha \in A\}$. If each $f_\alpha(E)$ is dense in E_α then the projective limit is said to be ***reduced***.

Remark 1.4.3. *Given a family $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ of locally convex t.v.s. over $\mathbb{K}$ which is directed by topological embeddings (i.e. for any $\alpha, \beta \in A$ there exists $\gamma \in A$ s.t. $E_\gamma \subseteq E_\alpha$ and $E_\gamma \subseteq E_\beta$ with continuous embeddings) and such that the set $E := \bigcap_{\alpha\in A} E_\alpha$ is dense in each E_α, we denote by i_α the embedding of E into E_α. The directedness of the family induces a partial order on A making A directed, i.e. $\alpha \leq \beta$ if and only if $E_\beta \subseteq E_\alpha$. For any $\alpha \leq \beta$, let us denote by $i_{\alpha\beta}$ the continuous embedding of E_β in E_α. Then the set E endowed with the projective topology τ_{proj} w.r.t. the family $\{(E_\alpha, \tau_\alpha, i_\alpha) : \alpha \in A\}$*

is the reduced projective limit of $\{(E_\alpha,\tau_\alpha) : \alpha \in A\}$ w.r.t. the mappings $\{i_{\alpha\beta} : \alpha,\beta \in A, \alpha \leq \beta\}$ and $\{i_\alpha : \alpha \in A\}$. For convenience, in such cases, (E,τ_{proj}) is called just reduced projective limit of $\{(E_\alpha,\tau_\alpha) : \alpha \in A\}$ (omitting the maps as they are all natural embeddings).

Example II: The space of test functions

Let $\Omega\subseteq\mathbb{R}^d$ be open in the euclidean topology. The space of test functions $\mathcal{C}_c^\infty(\Omega)$, i.e. the space of all the functions belonging to $\mathcal{C}^\infty(\Omega)$ which have a compact support, can be constructed as a reduced projective limit of the kind introduced in Remark 1.4.3. Consider the index set

$$T := \{t := (t_1, t_2) : t_1 \in \mathbb{N}_0,\, t_2 \in \mathcal{C}^\infty(\Omega) \text{ with } t_2(x) \geq 1,\ \forall\, x \in \Omega\}$$

and for each $t \in T$, let us introduce the following norm on $\mathcal{C}_c^\infty(\Omega)$:

$$\|\varphi\|_t := \sup_{x\in\Omega}\left(t_2(x) \sum_{|\alpha|\leq t_1} |(D^\alpha\varphi)(x)| \right).$$

For each $t \in T$, let $\mathscr{D}_t(\Omega)$ be the completion of $\mathcal{C}_c^\infty(\Omega)$ w.r.t. $\|\cdot\|_t$ and denote by τ_t the topology induced by the norm $\|\cdot\|_t$. Then the family $\{(\mathscr{D}_t(\Omega),\tau_t,i_t) : t \in T\}$ is directed by topological embeddings, since for any $t := (t_1,t_2), s := (s_1,s_2) \in T$ we always have that $r := (t_1+s_1, t_2+s_2) \in T$ is such that $\mathscr{D}_r(\Omega) \subseteq \mathscr{D}_t(\Omega)$ and $\mathscr{D}_r(\Omega) \subseteq \mathscr{D}_s(\Omega)$. Moreover, we have that as sets

$$\mathcal{C}_c^\infty(\Omega) = \bigcap_{t\in T} \mathscr{D}_t(\Omega).$$

Hence, the space of test functions $\mathcal{C}_c^\infty(\Omega)$ endowed with the projective topology τ_{proj} w.r.t. the family $\{(\mathscr{D}_t(\Omega),\tau_t,i_t) : t \in T\}$, where (for each $t \in T$) i_t denotes the natural embedding of $\mathcal{C}_c^\infty(\Omega)$ into $\mathscr{D}_t(\Omega)$ is the reduced projective limit of the family $\{(\mathscr{D}_t(\Omega),\tau_t) : t \in T\}$.

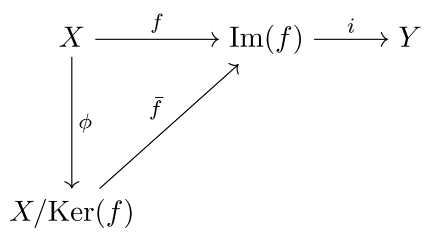
Using Sobolev embeddings theorems, it can be showed that the space of test functions $\mathcal{C}_c^\infty(\Omega)$ can be actually written as projective limit of a family of weighted Sobolev spaces which are Hilbert spaces (see [1, Chapter I, Section 3.10]).

1.5 Open mapping theorem

In this section we are going to come back for a moment to the general theory of metrizable t.v.s. to give one of the most celebrated theorems in this framework, the so-called open mapping theorem. Let us first try to motivate the question on which such a theorem is based on.

Let X and Y be two t.v.s. over $\mathbb{K}$ and $f : X \to Y$ a linear map. Then there exists a unique linear map $\bar{f} : X/\mathrm{Ker}(f) \to \mathrm{Im}(f)$ making the following diagram commutative, i.e.

$$\forall\, x \in X,\ f(x) = \bar{f}(\phi(x)). \tag{1.14}$$



where i is the natural injection of $\mathrm{Im}(f)$ into Y, i.e. the mapping which to each element y of $\mathrm{Im}(f)$ assigns that same element y regarded as an element of Y; ϕ is the canonical map of X onto its quotient $X/\mathrm{Ker}(f)$ (since we are between t.v.s. ϕ is continuous and open).

Note that

- $\bar{f}$ is well-defined.
  Indeed, if $\phi(x) = \phi(y)$, i.e. $x - y \in \mathrm{Ker}(f)$, then $f(x - y) = 0$ that is $f(x) = f(y)$ and so $\bar{f}(\phi(x)) = \bar{f}(\phi(y))$.
- $\bar{f}$ is linear.
  This is an immediate consequence of the linearity of f and of the linear structure of $X/\mathrm{Ker}(f)$.
- $\bar{f}$ is a one-to-one map of $X/\mathrm{Ker}(f)$ onto $\mathrm{Im}(f)$.
  The onto property is evident from the definition of $\mathrm{Im}(f)$ and of $\bar{f}$. As for the one-to-one property, note that $\bar{f}(\phi(x)) = \bar{f}(\phi(y))$ means by definition that $f(x) = f(y)$, i.e. $f(x - y) = 0$. This is equivalent, by linearity of f, to say that $x - y \in \mathrm{Ker}(f)$, which means that $\phi(x) = \phi(y)$.

Proposition 1.5.1. *Let $f : X \to Y$ a linear map between two t.v.s. X and Y. The map f is continuous if and only if the map $\bar{f}$ is continuous.*

Proof. Suppose f continuous and let U be an open subset in $\mathrm{Im}(f)$ (endowed with the subspace topology induced by the topology on Y). Then $f^{-1}(U)$ is open in X. By definition of $\bar{f}$, we have $\bar{f}^{-1}(U) = \phi(f^{-1}(U))$. Since the quotient map $\phi : X \to X/\mathrm{Ker}(f)$ is open, $\phi(f^{-1}(U))$ is open in $X/\mathrm{Ker}(f)$. Hence, $\bar{f}^{-1}(U)$ is open in $X/\mathrm{Ker}(f)$ and so the map $\bar{f}$ is continuous. Viceversa, suppose that $\bar{f}$ is continuous. Since $f = \bar{f} \circ \phi$ and ϕ is continuous, f is also continuous as composition of continuous maps. □

In general, the inverse of $\bar{f}$, which is well defined on $\mathrm{Im}(f)$ since $\bar{f}$ is injective, is not continuous, i.e. $\bar{f}$ is not necessarily open. However, combining the previous proposition with the definition of $\bar{f}$, it is easy to see that

Proposition 1.5.2. *Let $f : X \to Y$ a linear map between two t.v.s. X and Y. The map f is a topological homomorphism (i.e. linear, continuous and open) if and only if $\bar{f}$ is a topological isomorphism (i.e. bijective topological homomorphism).*

Now if Y is additionally Hausdorff and $\mathrm{Im}(f)$ finite dimensional, then whenever f is continuous we have that $\bar{f}$ is not only continuous but also open (see Theorem 3.1.1-c in TVS-I and recall that in this case $\mathrm{Ker}(f)$ is closed and so $X/\mathrm{Ker}(f)$ is a Hausdorff t.v.s..). Hence, any linear continuous map from a t.v.s. into a Hausdorff t.v.s. whose image is finite dimensional is also open. It is then natural to ask for which classes of t.v.s. any linear continuous map is also open. Of course, we are really interested in loosening the restriction of the finite dimensionality of $\mathrm{Im}(f)$ but we do expect that in doing so we shall give up some of the generality on the domain X of f. The open mapping theorem exactly provides an answer to this question.

Theorem 1.5.3 (Open Mapping Theorem)**.**
Let X and Y be two metrizable and complete t.v.s.. Every continuous linear surjective map $f : X \to Y$ is open.

The proof consists of two rather distinct parts. In the first one, we make use only of the fact that the mapping under consideration is onto and that Y is metrizable and complete (and so Baire). In the second part, we take advantage of the fact that both X and Y can be turned into metric spaces, and that Y is also complete.

Proof. Since Y is metrizable and complete, it is a Baire t.v.s. by Proposition 1.1.9. This together with the fact that $f : X \to Y$ is linear, continuous, onto map (and so $\mathrm{Im}(f)$ has non-empty interior) implies that the assumptions of Lemma 3.7 below are satisfied and so we get that $\overline{f(V)}$ is a neighbourhood of the origin in Y whenever V is a neighbourhood of the origin in X. This provides in particular that, for any $r > 0$ there exists $\rho > 0$ such that $B_\rho(o) \subseteq \overline{f(B_r(o))}$ since X and Y are both metrizable t.v.s.. Since the metrics employed can be always chosen to be translation invariant (see Proposition 1.1.3), we easily obtain that the assumption (1.15) in Lemma 3.8 below holds.

Let U be a neighbourhood of the origin in X. Then there exists $s > 0$ s.t. $B_s(o) \subseteq U$ and so $f(B_s(o)) \subseteq f(U)$. By applying (1.15) for $r = \frac{s}{2}$, we obtain that $\exists\ \rho := \rho_{\frac{s}{2}} > 0$ s.t. $B_\rho(o) \subseteq \overline{f(B_{\frac{s}{2}}(o))}$ and so, by Lemma 3.8, we have $B_\rho(o) \subseteq \overline{f(B_s(o))} \subseteq f(U)$ since $s > \frac{s}{2}$. Hence, $f(U)$ is a neighbourhood of the origin in Y. □

Lemma 1.5.4. *Let X be a t.v.s., Y a Baire t.v.s. and $f\colon X \to Y$ a continuous linear map. If $f(X)$ has non-empty interior, then $\overline{f(V)}$ is a neighbourhood of the origin in Y whenever V is a neighbourhood of the origin in X.*

Proof. (see Exercise Sheet 1) □

Lemma 1.5.5. *Let X be a metrizable and complete t.v.s. and Y a metrizable (not necessarily complete) t.v.s.. If $f\colon X \to Y$ is a continuous linear map such that*

$$\forall r > 0, \exists\ \rho_r > 0 \ \ s.t.\ B_{\rho_r}(f(x)) \subseteq \overline{f(B_r(x))}, \forall x \in X, \tag{1.15}$$

then for any $a > r$ we have that $B_{\rho_r}(f(x)) \subseteq f(B_a(x))$ for all $x \in X$.

Proof. Fixed $a > r > 0$, we can write $a = \sum_{n=0}^{\infty} r_n$ with $r_0 := r$ and $r_n > 0$ for all $n \in \mathbb{N}$. By assumption (1.15), we have that

$$\exists\ \rho_0 := \rho_{r_0} = \rho_r > 0 \ \text{ s.t. } B_{\rho_0}(f(x)) \subseteq \overline{f(B_{r_0}(x))}, \forall x \in X, \tag{1.16}$$

and

$$\forall n \in \mathbb{N}, \exists\ \rho_n := \rho_{r_n} > 0 \ \text{ s.t. } B_{\rho_n}(f(x)) \subseteq \overline{f(B_{r_n}(x))}, \forall x \in X. \tag{1.17}$$

W.l.o.g. we can assume that $(\rho_n)_{n\in\mathbb{N}}$ is strictly decreasing and convergent to zero.

Let $x \in X$ and $y \in B_{\rho_r}(f(x))$. We want to show that there exists a point $x' \in B_a(x)$ such that $y = f(x')$. To do that, we shall construct a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $(f(x_n))_{n\in\mathbb{N}}$ converges to y. Since X is complete, this will imply that $(x_n)_{n\in\mathbb{N}}$ converges to a point $x' \in X$, which necessarily satisfies $f(x') = y$ as f is continuous and Y Hausdorff. Of course, we need to define the sequence $(x_n)_{n\in\mathbb{N}}$ in such a way that its limit point x' lies in $B_a(x)$.

Since $y \in B_{\rho_r}(f(x)) \overset{(1.16)}{\subseteq} \overline{f(B_{r_0}(x))}$, there exists $x_1 \in B_{r_0}(x)$ such that $d_Y(f(x_1), y) < \rho_1$. Then $y \in B_{\rho_1}(f(x_1)) \overset{(1.17)}{\subseteq} \overline{f(B_{r_1}(x_1))}$ and so there exists

$x_2 \in B_{r_1}(x_1)$ such that $d_Y(f(x_2), y) < \rho_2$, i.e. $y \in B_{\rho_2}(f(x_2))$. By repeatedly applying (1.17), we get that for any $n \in \mathbb{N}$ there exists $x_{n+1} \in B_{r_n}(x_n)$ such that $d_Y(f(x_{n+1}), y) < \rho_{n+1}$. The sequence $(x_n)_{n\in\mathbb{N}}$ so obtained has all the desired properties. Indeed, for any $n \in \mathbb{N}$, we have $d_X(x_n, x_{n+1}) < r_n$ and so for any $m \geq l$ in $\mathbb{N}$ we get $d_X(x_l, x_m) \leq \sum_{j=l}^{m-1} d_X(x_j, x_{j+1}) < \sum_{j=l}^{\infty} r_j$, which implies that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Hence, by the completeness of X, there exists $x' \in X$ such that $d_X(x_n, x) \to 0$ as $n \to \infty$ and for any $n \in \mathbb{N}$ we get that

$$d_X(x, x') \leq d_X(x, x_1) + d_X(x_1, x_2) + \cdots + d_X(x_n, x') < \sum_{j=0}^{n-1} r_j + d_X(x_n, x').$$

Hence, $d_X(x, x') \leq \sum_{j=0}^{\infty} r_j = a$. Furthermore, for any $n \in \mathbb{N}$, we have $0 \leq d_Y(f(x_n), y) < \rho_n \to 0$, which implies the convergence of $(f(x_n))_{n\in\mathbb{N}}$ to y in Y. □

The Open Mapping Theorem 1.5.3 has several applications.

Corollary 1.5.6. *A bijective continuous linear map between two metrizable and complete t.v.s. is a topological isomorphism.*

Proof. Let X and Y be two metrizable and complete t.v.s. and $f : X \to Y$ bijective continuous and linear. Then, by the Open Mapping Theorem 1.5.3, we know that f is open, i.e. for any $U \in \mathcal{F}_X(o)$ we have that $f(U) \in \mathcal{F}_Y(o)$. This means that the inverse f^{-1}, whose existence is ensured by the injectivity of f, is continuous. □

Corollary 1.5.7. *A bijective linear map between two metrizable and complete t.v.s. with continuous inverse is continuous and so a topological isomorphism.*

Proof. Apply Corollary 1.5.6 to the inverse. □

Corollary 1.5.8. *Let τ_1 and τ_2 be two topologies on the same vector space X, both turning X into a metrizable complete t.v.s.. If τ_1 and τ_2 are comparable, then they coincide.*

Proof. Suppose that τ_1 is finer than τ_2. Then the identity map from (X, τ_1) to (X, τ_2) is bijective continuous and linear and so a topological isomorphism by Corollary 1.5.6. This means that also its inverse is continuous, i.e. the identity map from (X, τ_2) to (X, τ_1) is continuous. Hence, τ_2 is finer than τ_1. □

Corollary 1.5.9. *Let p and q be two norms on the same vector space X. If both (X,p) and (X,q) are Banach spaces and there exists $C>0$ such that $p(x)\leq Cq(x)$ for all $x\in X$, then the norms p and q are equivalent.*

Proof. Apply Corollary 1.5.8 to the topologies generated by p and by q. □

A fundamental result which can be derived from the Open Mapping Theorem 1.5.3 is the so called Closed Graph Theorem.

Theorem 1.5.10 (Closed Graph Theorem)**.**
Let X and Y be two metrizable and complete t.v.s.. Every linear map $f: X\to Y$ with closed graph is continuous.

Recall that the graph of a map $f: X\to Y$ is defined by

$$\mathrm{Gr}(f) := \{(x,y)\in X\times Y : y=f(x)\}.$$

The Closed Graph Theorem 1.5.10 will follow at once from the Open Mapping Theorem 1.5.3 and the following general result.

Proposition 1.5.11.
Let X and Y be two t.v.s. such that the following property holds.

$$\text{If } G \text{ is a closed linear subspace of } X\times Y \text{ and } g: G\to X \text{ is a continuous linear surjective map then } g \text{ is open.} \tag{1.18}$$

Then every linear map $f: X\to Y$ with closed graph is continuous.

Proof. Since X and Y are both t.v.s., $X\times Y$ endowed with the product topology is a t.v.s. and so the first and the second coordinate projections are both continuous. As $f: X\to Y$ is linear, $\mathrm{Gr}(f)$ is a linear subspace of $X\times Y$. Hence, $\mathrm{Gr}(f)$ endowed with subspace topology induced by the product topology, is itself a t.v.s. and the coordinate projections restricted to $\mathrm{Gr}(f)$, i.e.

$$p:\begin{array}{ccc}\mathrm{Gr}(f) & \to & X\\ (x,f(x)) & \mapsto & x\end{array} \quad \text{and} \quad q:\begin{array}{ccc}\mathrm{Gr}(f) & \to & Y\\ (x,f(x)) & \mapsto & f(x),\end{array}$$

are both continuous. Moreover, p is also linear and bijective, so there exists its inverse p^{-1} and we have that $f=q\circ p^{-1}$. Since p is a linear bijective and continuous map, (1.18) ensures that p is open, i.e. p^{-1} is continuous. Hence, f is continuous as composition of continuous maps. □

Proof. of Closed Graph Theorem
Since X and Y are both metrizable and complete t.v.s., (1.18) immediately follows from the Open Mapping Theorem 1.5.3. Indeed, if G is a closed linear subspace of $X\times Y$, then G endowed with the subspace topology induced by the product topology is also a metrizable and complete t.v.s. and so any $g : G \to X$ linear continuous and surjective is open by the Open Mapping Theorem 1.5.3. As (1.18) holds, we can apply Proposition 1.5.11, which ensures that every linear map $f : X \to Y$ with $\mathrm{Gr}(f)$ closed is continuous. □

The Closed Graph Theorem 1.5.10 and the Open Mapping Theorem 1.5.3 are actually equivalent, in the sense that we can also derive Theorem 1.5.3 from Theorem 1.5.10. To this purpose, we need to show a general topological result.

Proposition 1.5.12. *Let X and Y be two topological spaces such that Y is Hausdorff. Every continuous map from X to Y has closed graph.*

Proof. Let $f : X \to Y$ be continuous. We want to show that $(X\times Y)\setminus\mathrm{Gr}(f) := \{(x, y) \in X\times Y : y \neq f(x)\}$is open, i.e. for any $(x, y) \in (X\times Y)\setminus\mathrm{Gr}(f)$ we want to show that there exists a neighbourhood W of x in X and a neighbourhood U of y in Y such that $(x, y) \in W \times U \subseteq (X \times Y) \setminus \mathrm{Gr}(f)$.

As Y is Hausdorff and $y \neq f(x)$, there exist U neighbourhood of y in Y and V neighbourhood of $f(x)$ in Y such that $U \cap V = \emptyset$. The continuity of f guarantees that $f^{-1}(V)$ is a neighbourhood of x in X and so we have that $(x, y) \in f^{-1}(V) \times U$. We claim that $f^{-1}(V) \times U \subseteq (X \times Y) \setminus \mathrm{Gr}(f)$. If this was not the case, then there would exist $(\tilde{x}, \tilde{y}) \in f^{-1}(V) \times U$ such that $(\tilde{x}, \tilde{y}) \notin (X\times Y)\setminus\mathrm{Gr}(f)$. Hence, $\tilde{y} = f(\tilde{x}) \in f(f^{-1}(V)) \subseteq V$ and so $\tilde{y} \in U\cap V$ which yields a contradiction. □

Proof. of Open Mapping Theorem 1.5.3 using Closed Graph Theorem 1.5.10
Let f be a linear continuous and surjective map between two metrizable and complete t.v.s. X and Y. Then the map $\bar{f} : X/\mathrm{Ker}(f) \to Y$ defined in (1.14) is linear bijective and continuous by Proposition 1.5.1. Then Proposition 1.5.12 implies that $\mathrm{Gr}(\bar{f})$ is closed in $X/\mathrm{Ker}(f)\times Y$ endowed with the product topology. This gives in turn that the graph $\mathrm{Gr}(\bar{f}^{-1})$ of the inverse of $\bar{f}$ is closed in $Y \times X/\mathrm{Ker}(f)$, as $\mathrm{Gr}(\bar{f}^{-1}) = j(\mathrm{Gr}(\bar{f})$ where $j : X/\mathrm{Ker}(f) \times Y \to Y \times X/\mathrm{Ker}(f)$ is the homeomorphism given by $j(a, b) = (b, a)$. Hence, $\bar{f}^{-1}$ is a linear map with closed graph and so it is continuous by the Closed Graph Theorem 1.5.10. This means that $\bar{f}$ is open and so for any U neighbourhood of the origin in X we have $f(U) = \bar{f}(\phi(U))$ is open, i.e. f is open. □

The main advantage of the Closed Graph Theorem is that in many situations it is easier to prove that the graph of a map is closed rather than showing its continuity directly. For instance, we have seen that the inverse of an injective linear function with closed graph has also closed graph or that the inverse of a linear injective continuous map with Hausdorff codomain has closed graph. In both these cases, when we are in the realm of metrizable and complete t.v.s., we can conclude the continuity of the inverse thanks to the Closed Graph Theorem.

Chapter 2

Bounded subsets of topological vector spaces

In this chapter we will study the notion of bounded set in any t.v.s. and analyzing some properties which will be useful in the following and especially in relation with duality theory. Since compactness plays an important role in the theory of bounded sets, we will start this chapter by recalling some basic definitions and properties of compact subsets of a t.v.s..

2.1 Preliminaries on compactness

Let us recall some basic definitions of compact subset of a topological space (not necessarily a t.v.s.)

Definition 2.1.1. *A topological space X is said to be* compact *if X is Hausdorff and if every open covering $\{\Omega_i\}_{i\in I}$ of X contains a finite subcovering, i.e. for any collection $\{\Omega_i\}_{i\in I}$ of open subsets of X s.t. $\bigcup_{i\in I}\Omega_i = X$ there exists a finite subset $J\subseteq I$ s.t. $\bigcup_{j\in J}\Omega_j = X$.*

By going to the complements, we obtain the following equivalent definition of compactness.

Definition 2.1.2. *A topological space X is said to be* compact *if X is Hausdorff and if every family $\{F_i\}_{i\in I}$ of closed subsets of X whose intersection is empty contains a finite subfamily whose intersection is also empty, i.e. for any collection $\{F_i\}_{i\in I}$ of closed subsets of X s.t. $\bigcap_{i\in I}F_i = \emptyset$ there exists a finite subset $J\subseteq I$ s.t. $\bigcap_{j\in J}F_j = \emptyset$.*

Definition 2.1.3. *A subset K of a topological space X is said to be* compact *if K endowed with the topology induced by X is Hausdorff and for any collection $\{\Omega_i\}_{i\in I}$ of open subsets of X s.t. $\bigcup_{i\in I}\Omega_i \supseteq K$ there exists a finite subset $J\subseteq I$ s.t. $\bigcup_{j\in J}\Omega_j \supseteq K$.*

Let us state without proof a few well-known properties of compact spaces.

Proposition 2.1.4.

a) A closed subset of a compact space is compact.

b) Finite unions of compact sets are compact.

c) Let f be a continuous mapping of a compact space X into a Hausdorff topological space Y. Then $f(X)$ is a compact subset of Y.

In the following we will almost always be concerned with compact subsets of a Hausdorff t.v.s. X carrying the topology induced by X (and so which are themselves Hausdorff t.v.s.). Therefore, we now introduce a useful characterization of compactness in Hausdorff topological spaces.

Theorem 2.1.5. *Let X be a Hausdorff topological space. X is compact if and only if every filter on X has at least one accumulation point (see Definition 1.3.8).*

Proof.
Suppose that X is compact. Let $\mathcal{F}$ be a filter on X and $\mathcal{C} := \{\overline{M} : M \in \mathcal{F}\}$. As $\mathcal{F}$ is a filter, no finite intersection of elements in $\mathcal{C}$ can be empty. Therefore, by compactness, the intersection of all elements in $\mathcal{C}$ cannot be empty. Then there exists at least a point $x \in \overline{M}$ for all $M \in \mathcal{F}$, i.e. x is an accumulation point of $\mathcal{F}$. Conversely, suppose that every filter on X has at least one accumulation point. Let ϕ be a family of closed subsets of X whose intersection is empty. To show that X is compact, we need to show that there exists a finite subfamily of ϕ whose intersection is empty. Suppose by contradiction that no finite subfamily of ϕ has empty intersection. Then the family ϕ' of all the finite intersections of subsets belonging to ϕ forms a basis of a filter $\mathcal{F}$ on X. By our initial assumption, $\mathcal{F}$ has an accumulation point, say x. Thus, x belongs to the closure of any element of $\mathcal{F}$ and in particular to any set belonging to ϕ' (as the elements in ϕ' clearly belong to $\mathcal{F}$ and are closed). This means that x belongs to the intersection of all the sets belonging to ϕ', which is the same as the intersection of all the sets belonging to ϕ. But we had assumed the latter to be empty and so we have a contradiction. □

Corollary 2.1.6.
Any compact subset of a Hausdorff topological space is closed.

Proof.
Let K be a compact subset of a Hausdorff topological space X and let $x \in \overline{K}$. Denote by $\mathcal{F}(x)$ the filter of neighbourhoods of x in X and by $\mathcal{F}(x) \restriction K$ the filter in K generated by all the sets $U \cap K$ where $U \in \mathcal{F}(x)$. By Theorem 2.1.5, $\mathcal{F}(x) \restriction K$ has an accumulation point $x_1 \in K$. We claim that $x_1 \equiv x$, which implies that $\overline{K} = K$ and so that K is closed. In fact, if $x_1 \neq x$ then

the Hausdorffness of X implies that there exists $U \in \mathcal{F}(x)$ s.t. $X \setminus U$ is a neighbourhood of x_1 and, thus, $x_1 \notin \overline{U \cap K}$, which contradicts the fact that x_1 is an accumulation point of $\mathcal{F}(x) \restriction K$. □

Corollary 2.1.7.

1) Arbitrary intersections of compact subsets of a Hausdorff topological space are compact.

2) Any bijective continuous map from a compact space to a Hausdorff space is a homeomorphism.

3) Let τ_1, τ_2 be two Hausdorff topologies on a set X. If $\tau_1 \subseteq \tau_2$ and (X, τ_2) is compact then $\tau_1 \equiv \tau_2$.

Proof.

1. Let X be a Hausdorff topological space and $\{K_i\}_{i\in I}$ be an arbitrary family of compact subsets of X. Then each K_i is closed in X by Corollary 2.1.6 and so $\bigcap_{i\in I} K_i$ is a closed subset of each fixed K_i. As K_i is compact, Proposition 2.1.4-a) ensures that $\bigcap_{i\in I} K_i$ is compact in K_i and so in X.
2. Let U be an open subset of a compact space X and f a continuous map from X to a Hausdorff space Y. Since $X \setminus U$ is closed in X and X is compact, we have that $X \setminus U$ is compact in X by Proposition 2.1.4-a). Then Proposition 2.1.4-c) guarantees that $f(X \setminus U)$ is compact in Y, which implies in turn that $f(X \setminus U)$ is closed in Y by Corollary 2.1.6. Since f is bijective, we have that $Y \setminus f(U) = f(X \setminus U)$ and so that $f(U)$ is open. Hence, f^{-1} is continuous.
3. Since $\tau_1 \subseteq \tau_2$, the identity map from (X, τ_2) to (X, τ_1) is continuous and clearly bijective. Then the previous item implies that the identity from (X, τ_1) to (X, τ_2) is also continuous. Hence, $\tau_1 \equiv \tau_2$.

□

Last but not least, let us recall the following two definitions.

Definition 2.1.8. *A subset A of a topological space X is said to be* relatively compact *if the closure $\overline{A}$ of A is compact in X.*

Definition 2.1.9. *A subset A of a Hausdorff t.v.s. E is said to be precompact if A is relatively compact when viewed as a subset of the completion $\hat{E}$ of E.*

2.2 Bounded subsets: definition and general properties

Definition 2.2.1. *A subset B of a t.v.s. E is said to be* bounded *if for every U neighbourhood of the origin in E there exists $\lambda > 0$ such that $B \subseteq \lambda U$.*

In rough words this means that a subset B of E is bounded if B can be swallowed by any neighbourhood of the origin.

Proposition 2.2.2.

1. *If every element in some basis of neighbourhoods of the origin of a t.v.s. swallows a subset, then such a subset is bounded.*
2. *The closure of a bounded set is bounded.*
3. *Finite unions of bounded sets are bounded sets.*
4. *Any subset of a bounded set is a bounded set.*

Proof. Let E be a t.v.s. and $B \subset E$.

1. Suppose that $\mathcal{N}$ is a basis of neighbourhoods of the origin o in E such that for every $N \in \mathcal{N}$ there exists $\lambda_N > 0$ with $B \subseteq \lambda_N N$. Then, by definition of basis of neighbourhoods of o, for every U neighbourhood of o in E there exists $M \in \mathcal{N}$ s.t. $M \subseteq U$. Hence, there exists $\lambda_M > 0$ s.t. $B \subseteq \lambda_M M \subseteq \lambda_M U$, i.e. B is bounded.
2. Suppose that B is bounded in E. Then, as there always exists a basis $\mathcal{C}$ of neighbourhoods of the origin in E consisting of closed sets (see Corollary 2.1.14-a) in TVS-I), we have that for any $C \in \mathcal{C}$ there exists $\lambda > 0$ s.t. $B \subseteq \lambda C$ and thus $\overline{B} \subseteq \overline{\lambda C} = \lambda \overline{C} = \lambda C$. By Proposition 2.2.2-1, this is enough to conclude that $\overline{B}$ is bounded in E.
3. Let $n \in \mathbb{N}$ and $B_1, \ldots, B_n$ bounded subsets of E. As there always exists a basis $\mathcal{B}$ of balanced neighbourhoods of the origin in E (see Corollary 2.1.14-b) in TVS-I), we have that for any $V \in \mathcal{B}$ there exist $\lambda_1, \ldots, \lambda_n > 0$ s.t. $B_i \subseteq \lambda_i V$ for all $i = 1, \ldots, n$. Then $\bigcup_{i=1}^n B_i \subseteq \bigcup_{i=1}^n \lambda_i V \subseteq \left(\max_{i=1,\ldots,n} \lambda_i\right) V$, which implies the boundedness of $\bigcup_{i=1}^n B_i$ by Proposition 2.2.2-1.
4. Let B be bounded in E and let A be a subset of B. The boundedness of B guarantees that for any neighbourhood U of the origin in E there exists $\lambda > 0$ s.t. λU contains B and so A. Hence, A is bounded.

□

The properties in Proposition 2.2.2 lead to the following definition which is dually corresponding to the notion of basis of neighbourhoods.

Definition 2.2.3. *Let E be a t.v.s.. A family $\{B_\alpha\}_{\alpha\in I}$ of bounded subsets of E is called a* basis of bounded subsets *of E if for every bounded subset B of E there is $\alpha \in I$ s.t. $B \subseteq B_\alpha$.*

This duality between neighbourhoods and bounded subsets will play an important role in the study of the strong topology on the dual of a t.v.s.

Which sets do we know to be bounded in any t.v.s.?

- Singletons are bounded in any t.v.s., as every neighbourhood of the origin is absorbing.
- Finite subsets in any t.v.s. are bounded as finite union of singletons.

Proposition 2.2.4. *Compact subsets of a t.v.s. are bounded.*

Proof. Let E be a t.v.s. and K be a compact subset of E. For any neighbourhood U of the origin in E we can always find an open and balanced neighbourhood V of the origin s.t. $V \subseteq U$. Then we have

$$K \subseteq E = \bigcup_{n=0}^{\infty} nV.$$

From the compactness of K, it follows that there exist finitely many integers $n_1, \ldots, n_r \in \mathbb{N}_0$ s.t.

$$K \subseteq \bigcup_{i=1}^{r} n_i V \subseteq \left(\max_{i=1,\ldots,r} n_i\right) V \subseteq \left(\max_{i=1,\ldots,r} n_i\right) U.$$

Hence, K is bounded in E. □

This together with Corollary 2.1.6 gives that in any Hausdorff t.v.s. a compact subset is always bounded and closed. In finite dimensional Hausdorff t.v.s. we know that also the converse holds (because of Theorem 3.1.1 in TVS-I) and thus the ***Heine-Borel property*** always holds, i.e.

$$K \text{ compact} \Leftrightarrow K \text{ bounded and closed.}$$

This is not true, in general, in infinite dimensional t.v.s.

Example 2.2.5.
Let E be an infinite dimensional normed space. If every bounded and closed subset in E were compact, then in particular all the balls centered at the origin would be compact. Then the space E would be locally compact and so finite dimensional as proved in Theorem 3.2.1 in TVS-I, which gives a contradiction.

There is however an important class of infinite dimensional t.v.s., the so-called *Montel spaces*, in which the Heine-Borel property holds. Note that $\mathcal{C}^\infty(\mathbb{R}^d), \mathcal{C}_c^\infty(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d)$ are all Montel spaces.

Proposition 2.2.4 provides some further interesting classes of bounded subsets in a Hausdorff t.v.s..

Corollary 2.2.6. *Precompact subsets of a Hausdorff t.v.s. are bounded.*

Proof.
Let K be a precompact subset of a Hausdorff t.v.s. E and i the canonical embedding of E in its completion $\hat{E}$. By Definition 2.1.9, the closure of $i(K)$ in $\hat{E}$ is compact. Let U be any neighbourhood of the origin in E. Since i is a topological embedding, there is a neighbourhood $\hat{U}$ of the origin in $\hat{E}$ such that $U = \hat{U} \cap E$. Then, by Proposition 2.2.4, there is a number $\lambda > 0$ such that $\overline{i(K)} \subseteq \lambda\hat{U}$. Hence, we get

$$K \subseteq \overline{i(K)} \cap E \subseteq \lambda\hat{U} \cap E = \lambda\hat{U} \cap \lambda E = \lambda(\hat{U} \cap E) = \lambda U.$$ □

Corollary 2.2.7. *Let E be a Hausdorff t.v.s.. The union of a converging sequence in E and of its limit is a compact and so bounded closed subset in E.*

Proof. (Christmas assignment) □

Corollary 2.2.8. *Let E be a Hausdorff t.v.s.. Any Cauchy sequence in E is bounded.*

Proof. By using Corollary 2.2.7, one can show that any Cauchy sequence S in E is a precompact subset of E. Then it follows by Corollary 2.2.6 that S is bounded in E. □

Note that a Cauchy sequence S in a Hausdorff t.v.s. E is not necessarily relatively compact in E. Indeed, if this were the case, then its closure in E would be compact and so, by Theorem 2.1.5, the filter associated to S would have an accumulation point $x \in E$. Hence, by Proposition 1.3.10 and Proposition 1.1.31 in TVS-I, we get $S \to x \in E$ which is not necessarily true unless E is complete.

Proposition 2.2.9. *The image of a bounded set under a continuous linear map between t.v.s. is a bounded set.*

Proof. Let E and F be two t.v.s., $f : E \to F$ be linear and continuous, and $B \subseteq E$ be bounded. Then for any neighbourhood V of the origin in F, $f^{-1}(V)$ is a neighbourhood of the origin in E. By the boundedness of B in E, it follows that there exists $\lambda > 0$ s.t. $B \subseteq \lambda f^{-1}(V)$ and, thus, $f(B) \subseteq \lambda V$. Hence, $f(B)$ is a bounded subset of F. □

Corollary 2.2.10. *Let L be a continuous linear functional on a t.v.s. E. If B is a bounded subset of E, then $\sup_{x \in B} |L(x)| < \infty$.*

Proof. By Proposition 2.2.9, we have that $L(B)$ is bounded in $\mathbb{K}$. Hence, there exists $\lambda > 0$ such that $L(B)$ is contained in the closed ball of radius λ centered at the origin, i.e. for all $x \in B$ we have $|L(x)| \leq \lambda$, which yields the conclusion. □

Let us now introduce a general characterization of bounded sets in terms of sequences.

Proposition 2.2.11. *Let E be any t.v.s.. A subset B of E is bounded if and only if every sequence contained in B is bounded in E.*

Proof. The necessity of the condition is obvious from Proposition 2.2.2-4. Let us prove its sufficiency. Suppose that every sequence contained in B is bounded in E. If B is unbounded, then there exists a neighbourhood U of the origin in E s.t. for all $\lambda > 0$ we have $B \not\subseteq \lambda U$. W.l.o.g. we can assume U balanced. Then

$$\forall\, n \in \mathbb{N},\ \exists\, x_n \in B \text{ s.t. } x_n \notin nU. \tag{2.1}$$

By assumption the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded and so there exists $\mu > 0$ s.t. $\{x_n\}_{n\in\mathbb{N}} \subseteq \mu U$. Hence, there exists $m \in \mathbb{N}$ with $m \geq \mu$ such that $\{x_n\}_{n\in\mathbb{N}} \subseteq mU$ and in particular $x_m \in mU$, which contradicts (2.1). Hence, B must necessarily be bounded in E. □

2.3 Bounded subsets of special classes of t.v.s.

In this section we are going to study bounded sets in some of the special classes of t.v.s. which we have encountered so far. First of all, let us notice that any ball in a normed space is a bounded set and thus that in normed spaces there exist sets which are at the same time bounded and neighbourhoods of the origin. This property is actually a characteristic of all normable Hausdorff locally convex t.v.s.. Recall that a t.v.s. E is said to be *normable* if its topology can be defined by a norm, i.e. if there exists a norm $\|\cdot\|$ on E such that the collection $\{B_r : r > 0\}$ with $B_r := \{x \in E : \|x\| < r\}$ is a basis of neighbourhoods of the origin in E.

Proposition 2.3.1. *Let E be a Hausdorff locally convex t.v.s.. If there is a neighbourhood of the origin in E which is also bounded, then E is normable.*

Proof. Let U be a bounded neighbourhood of the origin in E. As E is locally convex, by Proposition 4.1.12 in TVS-I, we may always assume that U is open and absolutely convex, i.e. convex and balanced. The boundedness of U implies that for any balanced neighbourhood V of the origin in E there exists $\lambda > 0$ s.t. $U \subseteq \lambda V$. Hence, $U \subseteq nV$ for all $n \in \mathbb{N}$ such that $n \geq \lambda$, i.e. $\frac{1}{n}U \subseteq V$. Then the collection $\left\{\frac{1}{n}U\right\}_{n\in\mathbb{N}}$ is a basis of neighbourhoods of the origin o in E and, since E is a Hausdorff t.v.s., Corollary 2.2.4 in TVS-I guarantees that

$$\bigcap_{n\in\mathbb{N}} \frac{1}{n}U = \{o\}. \tag{2.2}$$

Since E is locally convex and U is an open absolutely convex neighbourhood of the origin, there exists a generating seminorm p on E s.t. $U = \{x \in E : p(x) < 1\}$ (see second part of proof of Theorem 4.2.9 in TVS-I). Then p must be a norm, because $p(x) = 0$ implies $x \in \frac{1}{n}U$ for all $n \in \mathbb{N}$ and so $x = 0$ by (2.2). Hence, E is normable. □

An interesting consequence of this result is the following one.

Corollary 2.3.2. *Let E be a locally convex metrizable space. If E is not normable, then E cannot have a countable basis of bounded sets in E.*

Proof. (Exercise Sheet 5) □

The notion of boundedness can be extended to linear maps between t.v.s..

Definition 2.3.3. *Let E, F be two t.v.s. and f a linear map of E into F. f is said to be* bounded *if for every bounded subset B of E, $f(B)$ is a bounded subset of F.*

We have already showed in Proposition 2.2.9 that any continuous linear map between two t.v.s. is a bounded map. The converse is not true in general but it holds for two special classes of t.v.s.: metrizable t.v.s. and LF-spaces.

Proposition 2.3.4. *Let E be a metrizable t.v.s. and let f be a linear map of E into a t.v.s. F. If f is bounded, then f is continuous.*

Proof. Let $f : E \to F$ be a bounded linear map. Suppose that f is not continuous. Then there exists a neighbourhood V of the origin in F whose preimage $f^{-1}(V)$ is not a neighbourhood of the origin in E. W.l.o.g. we can always assume that V is balanced. As E is metrizable, we can take a countable basis $\{U_n\}_{n\in\mathbb{N}}$ of neighbourhood of the origin in E s.t. $U_n \supseteq U_{n+1}$ for all $n \in \mathbb{N}$. Then for all $m \in \mathbb{N}$ we have $\frac{1}{m}U_m \not\subseteq f^{-1}(V)$ i.e.

$$\forall\, m \in \mathbb{N},\, \exists\, x_m \in \frac{1}{m}U_m \text{ s.t. } f(x_m) \notin V. \tag{2.3}$$

As for all $m \in \mathbb{N}$ we have $mx_m \in U_m$ we get that the sequence $\{mx_m\}_{m\in\mathbb{N}}$ converges to the origin o in E. In fact, for any neighbourhood U of the origin o in E there exists $\bar{n} \in \mathbb{N}$ s.t. $U_{\bar{n}} \subseteq U$. Then for all $n \geq \bar{n}$ we have $nx_n \in U_n \subseteq U_{\bar{n}} \subseteq U$, i.e. $\{mx_m\}_{m\in\mathbb{N}}$ converges to o.

Hence, Proposition 2.2.7 implies that $\{mx_m\}_{m\in\mathbb{N}_0}$ is bounded in E and so, since f is bounded, also $\{mf(x_m)\}_{m\in\mathbb{N}_0}$ is bounded in F. This means that there exists $\rho > 0$ s.t. $\{mf(x_m)\}_{m\in\mathbb{N}_0} \subseteq \rho V$. Then for all $n \in \mathbb{N}$ with $n \geq \rho$ we have $f(x_n) \in \frac{\rho}{n}V \subseteq V$ which contradicts (2.3).

□

To show that the previous proposition also hold for LF-spaces, we need to introduce the following characterization of bounded sets in LF-spaces.

Proposition 2.3.5.
Let (E, τ_{ind}) be an LF-space with defining sequence $\{(E_n, \tau_n)\}_{n\in\mathbb{N}}$. A subset B of E is bounded in E if and only if there exists $n \in \mathbb{N}$ s.t. B is contained in E_n and B is bounded in E_n.

To prove this result we will need the following refined version of Lemma 1.3.4.

Lemma 2.3.6. *Let Y be a locally convex space, Y_0 a closed linear subspace of Y equipped with the subspace topology, U a convex neighbourhood of the origin in Y_0, and $x_0 \in Y$ with $x_0 \notin U$. Then there exists a convex neighbourhood V of the origin in Y such that $x_0 \notin V$ and $V \cap Y_0 = U$.*

Proof.
By Lemma 1.3.4 we have that there exists a convex neighbourhood W of the origin in Y such that $W \cap Y_0 = U$. Now we need to distinguish two cases:
-If $x_0 \in Y_0$ then necessarily $x_0 \notin W$ since by assumption $x_0 \notin U$. Hence, we are done by taking $V := W$.
-If $x_0 \notin Y_0$, then let us consider the quotient Y/Y_0 and the canonical map $\phi : Y \to Y/Y_0$. As Y_0 is a closed linear subspace of Y and Y is locally convex,

we have that Y/Y_0 is Hausdorff and locally convex. Then, since $\phi(x_0) \neq o$, there exists a convex neighbourhood N of the origin o in Y/Y_0 such that $\phi(x_0) \notin N$. Set $\Omega := \phi^{-1}(N)$. Then Ω is a convex neighbourhood of the origin in Y such that $x_0 \notin \Omega$ and clearly $Y_0 \subseteq \Omega$ (as $\phi(Y_0) = o \in N$). Therefore, if we consider $V := \Omega \cap W$ then we have that: V is a convex neighbourhood of the origin in Y, $V \cap Y_0 = \Omega \cap W \cap Y_0 = W \cap Y_0 = U$ and $x_0 \notin V$ since $x_0 \notin \Omega$. □

Proof. of Proposition 2.3.5

Suppose first that B is contained and bounded in some E_n. Let U be an arbitrary neighbourhood of the origin in E. Then by Proposition 1.3.5 we have that $U_n := U \cap E_n$ is a neighbourhood of the origin in E_n. Since B is bounded in E_n, there is a number $\lambda > 0$ such that $B \subseteq \lambda U_n \subseteq \lambda U$, i.e. B is bounded in E.

Conversely, assume that B is bounded in E. Suppose that B is not contained in any of the E_n's, i.e. $\forall\, n \in \mathbb{N}$, $\exists\, x_n \in B$ s.t. $x_n \notin E_n$. We will show that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is not bounded in E and so a fortiori B cannot be bounded in E.

Since $x_1 \notin E_1$ but $x_1 \in B \subseteq E$ and E_1 is a closed linear subspace of (E, τ_{ind}), given an arbitrary convex neighbourhood U_1 of the origin in E_1 we can apply Lemma 2.3.6 and get that there exists V_2 convex neighbourhood of the origin in E s.t. $x_1 \notin V_2$ and $V_2 \cap E_1 = U_1$. As $\tau_{ind} \restriction E_2 = \tau_2$, we have that $U_2 := V_2 \cap E_2$ is a convex neighbourhood of the origin in E_2 s.t. $x_1 \notin U_2$ and $U_2 \cap E_1 = V_2 \cap E_2 \cap E_1 = V_2 \cap E_1 = U_1$.

Since $x_1 \notin U_2$, we can once again apply Lemma 2.3.6 and proceed as above to get that there exists U_3' convex neighbourhood of the origin in E_3 s.t. $x_1 \notin U_3'$ and $U_3' \cap E_2 = U_2$. Since $x_2 \notin E_2$ we also have that $\frac{1}{2}x_2 \notin E_2$ and so $\frac{1}{2}x_2 \notin U_2$. By applying again Lemma 2.3.6 and proceeding as above, we get that there exists U_3'' convex neighbourhood of the origin in E_3 s.t. $\frac{1}{2}x_2 \notin U_3''$ and $U_3'' \cap E_2 = U_2$. Taking $U_3 := U_3' \cap U_3''$ we have that $U_3 \cap E_2 = U_2$ and $x_1, \frac{1}{2}x_2 \notin U_2$.

By induction on n, we get a sequence $\{U_n\}_{n\in\mathbb{N}}$ such that for any $n \in \mathbb{N}$:

- U_n is a convex neighbourhood of the origin in E_n
- $U_n = U_{n+1} \cap E_n$ (and so $U_n \subseteq U_{n+1}$)
- $x_1, \frac{1}{2}x_2, \ldots, \frac{1}{n}x_n \notin U_{n+1}$.

Note that:

$$U_n = U_{n+1} \cap E_n = U_{n+2} \cap E_{n+1} \cap E_n = U_{n+2} \cap E_n = \cdots = U_{n+k} \cap E_n, \quad \forall k \in \mathbb{N}.$$

Consider $U := \bigcup_{j=1}^{\infty} U_j$, then for each $n \in \mathbb{N}$ we have

$$U \cap E_n = \left(\bigcup_{j=1}^{n} U_j \cap E_n \right) \cup \left(\bigcup_{j=n+1}^{\infty} U_j \cap E_n \right) = U_n \cup \left(\bigcup_{k=1}^{\infty} U_{n+k} \cap E_n \right) = U_n,$$

i.e. U is a neighbourhood of the origin in (E, τ_{ind}).

Suppose that $\{x_j\}_{j\in\mathbb{N}}$ is bounded in E and take a balanced neighbourhood V of the origin in E s.t. $V \subseteq U$. Then there exists $\lambda > 0$ s.t. $\{x_j\}_{j\in\mathbb{N}} \subseteq \lambda V$ and so $\{x_j\}_{j\in\mathbb{N}} \subseteq nV$ for all $n \in \mathbb{N}$ with $n \geq \lambda$. In particular, we have $x_n \in nV$ and so $\frac{1n}{x\ n} \in V \subseteq U$, which contradicts the third property of the U_j's (i.e. $1nx_n \notin= \bigcup_{j=1}^{\infty} U_{n+j}.\bigcup_{j=n+1}^{\infty} U_j = U$ since $U_j \subseteq U_{j+1}$ for all $j \in \mathbb{N}$). Hence, $\{x_j\}_{j\in\mathbb{N}}$ is not bounded in E and so B is not bounded in E. This contradicts our original assumption and so proves that $B \subseteq E_n$ for some $n \in \mathbb{N}$.

It remains to show that B is bounded in E_n. Let W_n be a neighbourhood of the origin in E_n. By Proposition 1.3.5, there exists a neighbourhood W of the origin in E such that $W \cap E_n = W_n$. Since B is bounded in E, there exists $\mu > 0$ s.t. $B \subseteq \mu W$ and hence

$$B = B \cap E_n \subseteq \mu W \cap E_n = \mu(W \cap E_n) = \mu W_n.$$

□

Corollary 2.3.7. *A bounded linear map from an LF- space into an arbitrary t.v.s. is always continuous.*

Proof. (Exercise Sheet 5) □

Chapter 3

Topologies on the dual space of a t.v.s.

In this chapter we are going to describe a general method to construct a whole class of topologies on the topological dual of a t.v.s. using the notion of polar of a subset. Among these topologies, usually called polar topologies, there are: the weak topology, the topology of compact convergence and the strong topology. In this chapter we will denote by:

- E a t.v.s. over the field $\mathbb{K}$ of real or complex numbers.
- E^* the algebraic dual of E, i.e. the vector space of all linear functionals on E.
- E' its topological dual of E, i.e. the vector space of all continuous linear functionals on E.

Moreover, given $x' \in E'$, we denote by $\langle x', x\rangle$ its value at the point x of E, i.e. $\langle x', x\rangle = x'(x)$. The bracket $\langle \cdot, \cdot \rangle$ is often called *pairing* between E and E'.

3.1 The polar of a subset of a t.v.s.

Definition 3.1.1. *Let A be a subset of E. We define the* polar of A *to be the subset A° of E' given by:*

$$A^\circ := \left\{ x' \in E' : \sup_{x \in A} |\langle x', x\rangle| \leq 1 \right\}.$$

Let us list some properties of polars:

a) The polar A° of a subset A of E is a convex balanced subset of E'.
b) If $A \subseteq B \subseteq E$, then $B^\circ \subseteq A^\circ$.
c) $(\rho A)^\circ = (\frac{1}{\rho})A^\circ, \forall\, \rho > 0, \forall\, A \subseteq E$.
d) $(A \cup B)^\circ = A^\circ \cap B^\circ, \forall\, A, B \subseteq E$.
e) If A is a cone in E, then $A^\circ \equiv \{x' \in E' : \langle x', x\rangle = 0, \forall\, x \in A\}$ and A° is a linear subspace of E'. In particular, this property holds when A is a linear subspace of E and, in this case, A° is called the *orthogonal of A*.

Proof.
Let us show just property e) while the proof of a), b), c) and d) is left as an exercise for the reader. Suppose that A is a cone, i.e. $\forall\,\lambda > 0$, $\forall x \in A, \lambda x \in A$. Then $x' \in A^\circ$ implies that $|\langle x', x\rangle| \leq 1$ for all $x \in A$. Since A is a cone, we must also have $|\langle x', \lambda x\rangle| \leq 1$ for all $x \in A$ and all $\lambda > 0$. This means that $|\langle x', x\rangle| \leq \frac{1}{\lambda}$ for all $x \in A$ and all $\lambda > 0$, which clearly gives $\langle x', x\rangle = 0$ for all $x \in A$. Hence, $A^\circ \subseteq \{x' \in E' : \langle x', x\rangle = 0, \forall\, x \in A\}$. The other inclusion is trivial. In this case, it is easy to see that A° is a linear subspace of E'. Indeed: $\forall\, x', y' \in A^\circ$, $\forall\, x \in A$, $\forall\, \lambda, \mu \in \mathbb{K}$ we have

$$\langle \lambda x' + \mu y', x\rangle = \lambda\langle x', x\rangle + \mu\langle y', x\rangle = \lambda \cdot 0 + \mu \cdot 0 = 0.$$

□

Proposition 3.1.2. *Let E be a t.v.s.. If B is a bounded subset of E, then the polar B° of B is an absorbing subset of E'.*

Proof.
Let $x' \in E'$. As B is bounded in E, Corollary 2.2.10 guarantees that any continuous linear functional x' on E is bounded on B, i.e. there exists a constant $M(x') > 0$ such that $\sup_{x\in B} |\langle x', x\rangle| \leq M(x')$. This implies that for any $\lambda \in \mathbb{K}$ with $|\lambda| \leq \frac{1}{M(x')}$ we have $\lambda x' \in B^\circ$, since

$$\sup_{x\in B} |\langle \lambda x', x\rangle| = |\lambda| \sup_{x\in B} |\langle x', x\rangle| \leq \frac{1}{M(x')} \cdot M(x') = 1.$$

□

3.2 Polar topologies on the topological dual of a t.v.s.

We are ready to define an entire class of topologies on the dual E' of E, called *polar topologies*. Consider a family Σ of bounded subsets of E with the following two properties:

(P1) If $A, B \in \Sigma$, then $\exists\, C \in \Sigma$ s.t. $A \cup B \subseteq C$.

(P2) If $A \in \Sigma$ and $\lambda \in \mathbb{K}$, then $\exists\, B \in \Sigma$ s.t. $\lambda A \subseteq B$.

Let us denote by Σ° the family of the polars of the sets belonging to Σ, i.e.

$$\Sigma^\circ := \{A^\circ : \; A \in \Sigma\}.$$

Claim: Σ° is a basis of neighbourhoods of the origin for a locally convex topology on E' compatible with the linear structure.

Proof. of Claim.
By Property a) of polars and by Proposition 3.1.2, all elements of Σ° are convex balanced absorbing susbsets of E'. Also:

1. $\forall\, A^\circ, B^\circ \in \Sigma^\circ$, $\exists\, C^\circ \in \Sigma^\circ$ s.t. $C^\circ \subseteq A^\circ \cap B^\circ$.
Indeed, if A° and B° in Σ° are respectively the polars of A and B in Σ, then by (P1) there exists $C \in \Sigma$ s.t. $A \cup B \subseteq C$ and so, by properties b) and d) of polars, we get: $C^\circ \subseteq (A \cup B)^\circ = A^\circ \cap B^\circ$.
2. $\forall\, A^\circ \in \Sigma^\circ$, $\forall\, \rho > 0$, $\exists\, B^\circ \in \Sigma^\circ$ s.t. $B^\circ \subseteq \rho A^\circ$.
Indeed, if A° in Σ° is the polar of A, then by (P2) there exists $B \in \Sigma$ s.t. $\frac{1}{\rho}A \subseteq B$ and so, by properties b) and c) of polars, we get that $B^\circ \subseteq \left(\frac{1}{\rho}A\right)^\circ = \rho A^\circ$.

By Theorem 4.1.14 in TVS-I, there exists a unique locally convex topology on E' compatible with the linear structure and having Σ° as a basis of neighborhoods of the origin. $\square$

Definition 3.2.1. *Given a family Σ of bounded subsets of a t.v.s. E s.t. (P1) and (P2) hold, we call* $\Sigma-$topology *on E' the locally convex topology defined by taking, as a basis of neighborhoods of the origin in E', the family Σ° of the polars of the subsets that belong to Σ. We denote by E'_Σ the space E' endowed with the Σ-topology.*

It is easy to see from the definition that (Exercise Sheet 6):

- The $\Sigma-$topology on E' is generated by the following family of seminorms:

$$\{p_A :\ A \in \Sigma\}, \text{ where } p_A(x') := \sup_{x \in A} |\langle x', x\rangle|, \forall\, x' \in E'. \tag{3.1}$$

- Define for any $A \in \Sigma$ and $\varepsilon > 0$ the following subset of E':

$$W_\varepsilon(A) := \left\{ x' \in E' : \sup_{x \in A} |\langle x', x\rangle| \leq \varepsilon \right\}.$$

  The family $\mathcal{B} := \{W_\varepsilon(A) :\ A \in \Sigma, \varepsilon > 0\}$ is a basis of neighbourhoods of the origin for the $\Sigma-$topology on E'.

Let us introduce now some important examples of polar topologies.

The weak topology on E'

The *weak topology on E'* is the $\Sigma-$topology corresponding to the family Σ of all finite subsets of E and it is usually denoted by $\sigma(E', E)$ (this topology is often also referred with the name of *weak*-topology* or *weak dual topology*). We denote by E'_σ the space E' endowed with the topology $\sigma(E', E)$.

A basis of neighborhoods of $\sigma(E', E)$ is given by the family

$$\mathcal{B}_\sigma := \{W_\varepsilon(x_1, \ldots, x_r) : r \in \mathbb{N}, x_1, \ldots, x_r \in E, \varepsilon > 0\}$$

where

$$W_\varepsilon(x_1, \ldots, x_r) := \left\{x' \in E' : |\langle x', x_j\rangle| \leq \varepsilon, \, j = 1, \ldots, r\right\}. \tag{3.2}$$

The topology of compact convergence on E'

The *topology of compact convergence on* E' is the $\Sigma-$topology corresponding to the family Σ of all compact subsets of E and it is usually denoted by $c(E', E)$. We denote by E'_c the space E' endowed with the topology $c(E', E)$.

The strong topology on E'

The *strong topology on* E' is the $\Sigma-$topology corresponding to the family Σ of all bounded subsets of E and it is usually denoted by $b(E', E)$. As a filter in E' converges to the origin in the strong topology if and only if it converges to the origin uniformly on every bounded subset of E (see Proposition 3.2.2), the strong topology on E' is sometimes also referred as *the topology of bounded convergence*. When E' carries the strong topology, it is usually called the *strong dual of* E and denoted by E'_b.

Let us look now at some general properties of polar topologies and how they relate to the above examples.

Proposition 3.2.2. *A filter $\mathcal{F}'$ on E' converges to an element $x' \in E'$ in the Σ-topology on E' if and only if $\mathcal{F}'$ converges uniformly to x' on each subset A belonging to Σ, i.e. the following holds:*

$$\forall \varepsilon > 0, \forall A \in \Sigma, \; \exists \, M' \in \mathcal{F}' \; s.t. \; \sup_{x \in A} |\langle x', x\rangle - \langle y', x\rangle| \leq \varepsilon, \, \forall \, y' \in M'. \tag{3.3}$$

This proposition explains why the $\Sigma-$topology on E' is often referred as *topology of the uniform converge over the sets of* Σ.

Proof.

Suppose that (3.3) holds and let U be a neighbourhood of the origin in the $\Sigma-$topology on E'. Then there exists $\varepsilon > 0$ and $A \in \Sigma$ s.t. $W_\varepsilon(A) \subseteq U$ and so

$$x' + W_\varepsilon(A) \subseteq x' + U. \tag{3.4}$$

On the other hand, since we have that

$$\begin{aligned} x' + W_\varepsilon(A) &= \left\{ x' + y' \in E' : \sup_{x\in A} |\langle y', x\rangle| \leq \varepsilon \right\} \\ &= \left\{ z' \in E' : \sup_{x\in A} |\langle z' - x', x\rangle| \leq \varepsilon \right\}, \end{aligned} \tag{3.5}$$

the condition (3.3) together with (3.4) gives that

$$\exists\ M' \in \mathcal{F}' \text{ s.t. } M' \subseteq x' + W_\varepsilon(A) \subseteq x' + U.$$

The latter implies that $x' + U \in \mathcal{F}'$ since $\mathcal{F}'$ is a filter and so the family of all neighbourhoods of x' in the $\Sigma-$topology on E' is contained in $\mathcal{F}'$, i.e. $\mathcal{F}' \to x'$.

Conversely, if $\mathcal{F}' \to x'$, then for any neighbourhood V of x' in the $\Sigma-$topology on E' we have $V \in \mathcal{F}'$. In particular, for all $A \in \Sigma$ and for all $\varepsilon > 0$ we have $x' + W_\varepsilon(A) \in \mathcal{F}'$. Then by taking $M' := x' + W_\varepsilon(A)$ and using (3.5), we easily get (3.3). □

Remark 3.2.3. *Using the previous result, one can easily show that sequence $\{x'_n\}_{n\in\mathbb{N}}$ of elements in E' converges to the origin in the weak topology if and only if at each point $x \in E$ the sequence of their values $\{\langle x'_n, x\rangle\}_{n\in\mathbb{N}}$ converges to zero in $\mathbb{K}$ (see Exercise Sheet 6). In other words, the weak topology on E' is nothing else but the topology of pointwise convergence in E, when we look at continuous linear functionals on E simply as functions on E.*

In general we can compare two polar topologies by using the following criterion: *If Σ_1 and Σ_2 are two families of bounded subsets of a t.v.s. E such that (P1) and (P2) hold and $\Sigma_1 \supseteq \Sigma_2$, then the Σ_1-topology is finer than the Σ_2-topology.* In particular, this gives the following comparison relations between the three polar topologies on E' introduced above:

$$\sigma(E', E) \subseteq c(E', E) \subseteq b(E', E).$$

Proposition 3.2.4. *Let Σ be a family of bounded subsets of a t.v.s. E s.t. (P1) and (P2) hold. If the union of all subsets in Σ is dense in E, then E'_Σ is Hausdorff.*

Proof. Assume that the union of all subsets in Σ is dense in E. As the $\Sigma-$topology is locally convex, to show that E'_Σ is Hausdorff is enough to check that the family of seminorms in (3.1) is separating (see Proposition 4.3.3 in TVS-I). Suppose that $p_A(x') = 0$ for all $A \in \Sigma$, then

$$\sup_{x\in\ A} |\langle x', x\rangle| = 0,\ \forall A \in \Sigma,$$

which gives

$$\langle x', x\rangle = 0, \forall x \in \bigcup_{A\in\Sigma} A.$$

As the continuous functional x' is zero on a dense subset of E, it has to be identically zero on the whole E. Hence, the family $\{p_A : A \in \Sigma\}$ is a separating family of seminorms which generates the $\Sigma-$topology on E'. □

Corollary 3.2.5. *The topology of compact convergence, the weak and the strong topologies on E' are all Hausdorff.*

Let us consider now for any $x \in E$ the linear functional v_x on E' which associates to each element of the dual E' its *"value at the point x"*, i.e.

$$\begin{array}{rccl} v_x : & E' & \to & \mathbb{K} \\ & x' & \mapsto & \langle x', x\rangle. \end{array}$$

Clearly, each $v_x \in (E')^*$ but when can we say that $v_x \in (E'_\Sigma)'$? Can we find conditions on Σ which guarantee the continuity of v_x w.r.t. the $\Sigma-$topology?

Fixed an arbitrary $x \in E$, v_x is continuous on E'_Σ if and only if for any $\varepsilon > 0$, $v_x^{-1}(\bar{B}_\varepsilon(0))$ is a neighbourhood of the origin in E' w.r.t. the $\Sigma-$topology ($\bar{B}_\varepsilon(0)$ denotes the closed ball of radius ε and center 0 in $\mathbb{K}$). This means that

$$\forall\, \varepsilon > 0, \exists\, A \in \Sigma : \; A^\circ \subseteq v_x^{-1}(\bar{B}_\varepsilon(0)) = \{x' \in E' : |\langle x', x\rangle| \leq \varepsilon\}$$

i.e.

$$\forall\, \varepsilon > 0, \exists\, A \in \Sigma : \left|\langle x', \frac{1}{\varepsilon}x\rangle\right| \leq 1, \forall\, x' \in A^\circ. \tag{3.6}$$

Then it is easy to see that the following holds:

Proposition 3.2.6. *Let Σ be a family of bounded subsets of a t.v.s. E s.t. (P1) and (P2) hold. If Σ covers E then for every $x \in E$ the value at x is a continuous linear functional on E'_Σ, i.e. $v_x \in (E'_\Sigma)'$.*

Proof. If $E \subseteq \bigcup_{A\in\Sigma} A$ then for any $x \in E$ and any $\varepsilon > 0$ we have $\frac{1}{\varepsilon} \in A$ for some $A \in \Sigma$ and so $|\langle x', \frac{1}{\varepsilon}x\rangle| \leq 1$ for all $x' \in A^\circ$. This means that (3.6) holds, which is equivalent to v_x being continuous w.r.t. the $\Sigma-$topology on E'. □

The previous proposition is useful to get the following characterization of the weak topology on E', which is often taken as a definition for this topology.

Proposition 3.2.7. *Let E be a t.v.s.. The weak topology on E' is the coarsest topology on E' such that, for all $x \in E$, v_x is continuous.*

Proof.
Since the weak topology $\sigma(E', E)$ is by definition the $\Sigma-$topology on E' corresponding to the family Σ of all finite subsets of E which clearly covers E, Proposition 3.2.6 ensures that all v_x are continuous on E'_σ.[1] Moreover, if there would exist a topology τ on E' strictly coarser that $\sigma(E', E)$ and such that all v_x were continuous, then in particular $\forall\, \varepsilon > 0$, $\forall\, r \in \mathbb{N}$, $\forall\, x_1, \ldots, x_r \in E$, each $v_{x_i}^{-1}(\bar{B}_\varepsilon(0))$ would be a neighbourhood of the origin in (E', τ) for $i = 1, \ldots, r$. Hence, each $W_\varepsilon(x_1, \ldots, x_r)$ would be a neighbourhood of the origin in (E', τ), since $W_\varepsilon(x_1, \ldots, x_r) = \bigcap_{i=1}^r v_{x_i}^{-1}(\bar{B}_\varepsilon(0))$ (cf. (3.2)). Therefore, any element of a basis of neighborhoods of the origin in E'_σ is also a neighbourhood of the origin in (E', τ). This implies that the two topologies τ and $\sigma(E', E)$ must necessarily coincide. □

Proposition 3.2.6 means that, if Σ covers E then the image of E under the canonical map

$$\begin{array}{rccc} \varphi: & E & \to & (E'_\Sigma)^* \\ & x & \mapsto & v_x. \end{array}$$

is contained in the topological dual of E'_Σ, i.e. $\varphi(E) \subseteq (E'_\Sigma)'$. In general, the canonical map $\varphi : E \to (E'_\Sigma)'$ is neither injective or surjective. However, when we restrict our attention to locally convex Hausdorff t.v.s., the following consequence of Hahn-Banach theorem guarantees the injectivity of the canonical map.

Lemma 3.2.8. *If E is a locally convex Hausdorff t.v.s with $E \neq \{o\}$, then for every $o \neq x_0 \in E$ there exists $x' \in E'$ s.t. $\langle x', x_0 \rangle \neq 0$, i.e. $E' \neq \{o\}$.*

Proof. (see Interactive Sheet 3) □

Corollary 3.2.9. *Let E be a non-trivial locally convex Hausdorff t.v.s and Σ a family of bounded subsets of E s.t. (P1) and (P2) hold and Σ covers E. Then the canonical map $\varphi : E \to (E'_\Sigma)'$ is injective.*

Proof. Let $o \neq x_0 \in E$. By Proposition 3.2.8, we know that there exists $x' \in E'$ s.t. $v_x(x') \neq 0$ which proves that v_x is not identically zero on E' and so that $Ker(\varphi) = \{o\}$. Hence, φ is injective. □

[1] Fixed $x \in E$, one could also show the continuity of v_x w.r.t. $\sigma(E', E)$ by simply noticing that $|v_x(x')| = p_{\{x\}}(x')$ for any $x' \in E'$ and using Corollary 4.6.2. in TVS-I about continuity of functionals on locally convex t.v.s.

In the particular case of the weak topology on E' the canonical map $\varphi : E \to (E'_\sigma)'$ is also surjective, and so E can be regarded as the dual of its weak dual E'_σ. To show this result we will need to use the following consequence of Hahn-Banach theorem:

Lemma 3.2.10. *Let Y be a closed linear subspace of a locally convex t.v.s. X. If $Y \neq X$, then there exists $f \in X'$ s.t. f is not identically zero on X but identically vanishes on Y.*

Proof. (see Exercise Sheet 6)

Proposition 3.2.11. *Let E be a locally convex Hausdorff t.v.s. with $E \neq \{o\}$. Then the canonical map $\varphi : E \to (E'_\sigma)'$ is an isomorphism.*

Proof. Let $L \in (E'_\sigma)'$. By the definition of $\sigma(E', E)$ and Proposition 4.6.1 in TVS-I, we have that there exist $F \subset E$ with $|F| < \infty$ and $C > 0$ s.t.

$$|L(x')| \leq C p_F(x') = C \sup_{x \in F} |\langle x', x \rangle|. \tag{3.7}$$

Take $M := span(F)$ and $d := dim(M)$. Consider an algebraic basis $\mathcal{B} := \{e_1, \ldots, e_d\}$ of M and for each $j \in \{1, \ldots, d\}$ apply Lemma 3.2.10 to $Y := span\{\mathcal{B} \setminus \{e_j\}\}$ and $X := M$. Then for each $j \in \{1, \ldots, d\}$ there exists $f_j : M \to \mathbb{K}$ linear and continuous such that $\langle f_j, e_k \rangle = 0$ if $k \neq j$ and $\langle f_j, e_j \rangle \neq 0$. W.l.o.g. we can assume $\langle f_j, e_j \rangle = 1$. By applying the Hahn-Banach theorem (see Theorem 5.1.1 in TVS-I), we get that for each $j \in \{1, \ldots, d\}$ there exists $e'_j : E \to \mathbb{K}$ linear and continuous such that $e'_j \restriction_M = f_j$, in particular $\langle e'_j, e_k \rangle = 0$ for $k \neq j$ and $\langle e'_j, e_j \rangle = 1$.

Let $M' := span\{e'_1, \ldots, e'_d\} \subset E'$, $x_L := \sum_{j=1}^d L(e'_j) e_j \in M$ and for any $x' \in E'$ define $p(x') := \sum_{j=1}^d \langle x', e_j \rangle e'_j \in M'$. Then for any $x' \in E'$ we get that:

$$\langle x', x_L \rangle = \sum_{j=1}^d L(e'_j) \langle x', e_j \rangle = L(p(x')) \tag{3.8}$$

and also

$$\langle x' - p(x'), e_k \rangle = \langle x', e_k \rangle - \sum_{j=1}^d \langle x', e_j \rangle \langle e'_j, e_k \rangle = \langle x', e_k \rangle - \langle x', e_k \rangle \langle e_k, e_k \rangle = 0$$

which gives

$$\langle x' - p(x'), m \rangle = 0, \forall m \in M. \tag{3.9}$$

Then for all $x' \in E'$ we have:

$$|L(x' - p(x'))| \overset{(3.7)}{\leq} C \sup_{x \in F} |\langle x' - p(x'), x\rangle| \overset{(3.9)}{=} 0$$

which give that $L(x') = L(p(x')) \overset{(3.8)}{=} \langle x', x_L\rangle = v_{x_L}(x')$. Hence, we have proved that for every $L \in (E'_\sigma)'$ there exists $x_L \in E$ s.t. $\varphi(x_L) \equiv v_{x_L} \equiv L$, i.e. $\varphi : E \to (E'_\sigma)'$ is surjective. Then we are done because the injectivity of $\varphi : E \to (E'_\sigma)'$ follows by applying Corollary 3.2.9 to this special case. □

Remark 3.2.12. *The previous result suggests that it is indeed more convenient to restrict our attention to locally convex Hausdorff t.v.s. when dealing with weak duals. Moreover, as showed in Proposition 3.2.8, considering locally convex Hausdorff t.v.s has the advantage of avoiding the pathological situation in which the topological dual of a non-trivial t.v.s. is reduced to the only zero functional (for an example of a t.v.s. on which there are no continuous linear functional than the trivial one, see Exercise Sheet 6).*

3.3 The polar of a neighbourhood in a locally convex t.v.s.

Let us come back now to the study of the weak topology and prove one of the milestones of the t.v.s. theory: the *Banach-Alaoglu-Bourbaki theorem.* To prove this important result we need to look for a moment at the algebraic dual E^* of a t.v.s. E. In analogy to what we did in the previous section, we can define *the weak topology on the algebraic dual* E^* (which we will denote by $\sigma(E^*, E)$) as the coarsest topology such that for any $x \in E$ the linear functional w_x is continuous, where

$$\begin{array}{rccl} w_x : & E^* & \to & \mathbb{K} \\ & x^* & \mapsto & \langle x^*, x\rangle := x^*(x). \end{array} \tag{3.10}$$

(Note that $w_x \restriction E' = v_x$). Equivalently, the weak topology on the algebraic dual E^* is the locally convex topology on E^* generated by the family $\{q_F : F \subseteq E, |F| < \infty\}$ of seminorms $q_F(x^*) := \sup_{x \in F} |\langle x^*, x\rangle|$ on E^*. It is then easy to see that $\sigma(E', E) = \sigma(E^*, E) \restriction E'$.

An interesting property of the weak topology on the algebraic dual of a t.v.s. is the following one:

Proposition 3.3.1. *If E is a t.v.s. over $\mathbb{K}$, then its algebraic dual E^* endowed with the weak topology $\sigma(E^*, E)$ is topologically isomorphic to the product of $dim(E)$ copies of the field $\mathbb{K}$ endowed with the product topology.*

Proof.
Let $\{e_i\}_{i\in I}$ be an algebraic basis of E, i.e. $\forall\, x \in E$, $\exists\, \{x_i\}_{i\in I} \in \mathbb{K}^{dim(E)}$ s.t. $x = \sum_{i\in I} x_i e_i$. For any linear functions $L : E \to \mathbb{K}$ and any $x \in E$ we then have $L(x) = \sum_{i\in I} x_i L(e_i)$. Hence, L is completely determined by the sequence $\{L(e_i)\}_{i\in I} \in \mathbb{K}^{dim(E)}$. Conversely, every element $u := \{u_i\}_{i\in I} \in \mathbb{K}^{dim(E)}$ uniquely defines the linear functional L_u on E via $L_u(e_i) := u_i$ for all $i \in I$. This completes the proof that E^* is algebraically isomorphic to $\mathbb{K}^{dim(E)}$. Moreover, the collection $\{W_\varepsilon(e_{i_1}, \ldots, e_{i_r}) : \varepsilon > 0,\, r \in \mathbb{N},\, i_1, \ldots, i_r \in I\}$, where

$$W_\varepsilon(e_{i_1}, \ldots, e_{i_r}) := \{x^* \in E^* : |\langle x^*, e_{i_j}\rangle| \leq \varepsilon, \text{ for } j = 1, \ldots, r\},$$

is a basis of neighbourhoods of the origin in $(E^*, \sigma(E^*, E))$. Via the isomorphism described above, we have that for any $\varepsilon > 0$, $r \in \mathbb{N}$, and $i_1, \ldots, i_r \in I$:

$$\begin{aligned} W_\varepsilon(e_{i_1}, \ldots, e_{i_r}) &\approx \Big\{\{u_i\}_{i\in I} \in \mathbb{K}^{dim(E)} : |u_{i_j}| \leq \varepsilon, \text{ for } j = 1, \ldots, r\Big\} \\ &= \prod_{j=1}^{r} \bar{B}_\varepsilon(0) \times \prod_{I \setminus \{i_1, \ldots, i_r\}} \mathbb{K} \end{aligned}$$

and so $W_\varepsilon(e_{i_1}, \ldots, e_{i_r})$ is a neighbourhood of the product topology τ_{prod} on $\mathbb{K}^{dim(E)}$ (recall that we always consider the euclidean topology on $\mathbb{K}$). Therefore, $(E^*, \sigma(E^*, E))$ is topological isomorphic to $\left(\mathbb{K}^{dim(E)}, \tau_{prod}\right)$. □

Let us now focus our attention on the polar of a neighbourhood U of the origin in a non-trivial locally convex Hausdorff t.v.s. E. We are considering here only non-trivial locally convex Hausdorff t.v.s. in order to be sure to have non-trivial continuous linear functionals (see Remark 3.2.12) and so to make a meaningful analysis on the topological dual.

First of all let us observe that:

$$\{x^* \in E^* : \sup_{x\in U} |\langle x^*, x\rangle| \leq 1\} \equiv U^\circ := \{x' \in E' : \sup_{x\in U} |\langle x', x\rangle| \leq 1\}. \quad (3.11)$$

Indeed, since $E' \subseteq E^*$, we clearly have $U^\circ \subseteq \{x^* \in E^* : \sup_{x\in U} |\langle x^*, x\rangle| \leq 1\}$. Moreover, any linear functional $x^* \in E^*$ s.t. $\sup_{x\in U} |\langle x^*, x\rangle| \leq 1$ is continuous on E and it is therefore an element of E'.

It is then quite straightforward to show that:

Proposition 3.3.2. *The polar of a neighbourhood U of the origin in E is closed w.r.t. $\sigma(E^*, E)$.*

Proof. By (3.11) and (3.10), it is clear that $U^\circ = \bigcap_{x\in U} w_x^{-1}([-1,1])$. On the other hand, by definition of $\sigma(E^*, E)$ we have that w_x is continuous on $(E^*, \sigma(E^*, E))$ for all $x \in E$ and so each $w_x^{-1}([-1,1])$ is closed in $(E^*, \sigma(E^*, E))$. Hence, U° is closed in $(E^*, \sigma(E^*, E))$ as the intersection of closed subsets of $(E^*, \sigma(E^*, E))$. □

We are ready now to prove the famous Banach-Alaoglu-Bourbaki Theorem

Theorem 3.3.3 (Banach-Alaoglu-Bourbaki Theorem).
The polar of a neighbourhood U of the origin in a locally convex Hausdorff t.v.s. $E \neq \{o\}$ is compact in E'_σ.

Proof.
Since U is a neighbourhood of the origin in E, U is absorbing in E, i.e. $\forall\, x \in E$, $\exists\, M_x > 0$ s.t. $M_x x \in U$. Hence, for all $x \in E$ and all $x' \in U^\circ$ we have $|\langle x', M_x x\rangle| \leq 1$, which is equivalent to:

$$\forall\, x \in E,\ \forall\, x' \in U^\circ,\ \ |\langle x', x\rangle| \leq \frac{1}{M_x}. \tag{3.12}$$

Moreover, for any $x \in E$, the subset

$$D_x := \left\{ \alpha \in \mathbb{K} :\ |\alpha| \leq \frac{1}{M_x} \right\}$$

is compact in $\mathbb{K}$ w.r.t. to the euclidean topology.

Consider an algebraic basis $\mathcal{B}$ of E, then by Tychnoff's theorem[2] the subset $P := \prod_{x\in\mathcal{B}} D_x$ is compact in $\left(\mathbb{K}^{dim(E)}, \tau_{prod}\right)$.

Using the isomorphism introduced in Proposition 3.3.1 and (3.11), we get that

$$U^\circ \approx \{(\langle x^*, x\rangle)_{x\in\mathcal{B}} : x^* \in U^\circ\}$$

and so by (3.12) we have that $U^\circ \subset P$. Since Corollary 3.3.2 and Proposition 3.3.1 ensure that U° is closed in $\left(\mathbb{K}^{dim(E)}, \tau_{prod}\right)$, we get that U° is a closed subset of P. Hence, by Proposition 2.1.4–1, U° is compact $\left(\mathbb{K}^{dim(E)}, \tau_{prod}\right)$ and so in $(E^*, \sigma(E^*, E))$. As $U^\circ = E' \cap U^\circ$ we easily see that U° is compact in $(E', \sigma(E', E))$. □

[2]**Tychnoff's theorem**: The product of an arbitrary family of compact spaces endowed with the product topology is also compact.

We briefly introduce now a nice consequence of Banach-Alaoglu-Bourbaki theorem. Let us start by introducing a norm on the topological dual space E' of a seminormed space (E, ρ):

$$\rho'(x') := \sup_{x \in E : \rho(x) \leq 1} |\langle x', x \rangle|.$$

ρ' is usually called the *operator norm* on E'.

Corollary 3.3.4. *Let (E, ρ) be a non-trivial normed space. The closed unit ball in E' w.r.t. the operator norm ρ' is compact in E'_σ.*

Proof. First of all, let us note that a normed space it is indeed a locally convex Hausdorff t.v.s.. Then, by applying Banach-Alaoglu-Borubaki theorem to the closed unit ball $\bar{B}_1(o)$ in (E, ρ), we get that $\left(\bar{B}_1(o)\right)^\circ$ is compact in E'_σ. The conclusion then easily follow by the observation that $\left(\bar{B}_1(o)\right)^\circ$ actually coincides with the closed unit ball in (E', ρ'):

$$\begin{aligned}
\left(\bar{B}_1(o)\right)^\circ &= \{x' \in E' : \sup_{x \in \bar{B}_1(o)} |\langle x', x \rangle| \leq 1\} \\
&= \{x' \in E' : \sup_{x \in E', \rho(x) \leq 1} |\langle x', x \rangle| \leq 1\} \\
&= \{x' \in E' : \rho'(x') \leq 1\}.
\end{aligned}$$

□

Chapter 4

Tensor products of t.v.s.

4.1 Tensor product of vector spaces

As usual, we consider only vector spaces over the field $\mathbb{K}$ of real numbers or of complex numbers.

Definition 4.1.1.
*Let E, F, M be three vector spaces over $\mathbb{K}$ and $\phi : E \times F \to M$ be a bilinear map. E and F are said to be ϕ-*linearly disjoint *if:*

(LD) *For any $r \in \mathbb{N}$, any $\{x_1, \dots, x_r\}$ finite subset of E and any $\{y_1, \dots, y_r\}$ finite subset of F s.t. $\sum_{i=1}^{r} \phi(x_i, y_j) = 0$, we have that both the following conditions hold:*

- *if $x_1, \dots, x_r$ are linearly independent in E, then $y_1 = \dots = y_r = 0$*
- *if $y_1, \dots, y_r$ are linearly independent in F, then $x_1 = \dots = x_r = 0$*

Recall that, given three vector spaces over $\mathbb{K}$, a map $\phi : E \times F \to M$ is said to be *bilinear* if:

$$\forall\, x_0 \in E, \quad \phi_{x_0} : \begin{array}{ccc} F & \to & M \\ y & \to & \phi(x_0, y) \end{array} \qquad \text{is linear}$$

and

$$\forall\, y_0 \in F, \quad \phi_{y_0} : \begin{array}{ccc} E & \to & M \\ x & \to & \phi(x, y_0) \end{array} \qquad \text{is linear.}$$

Let us give a useful characterization of $\phi-$linear disjointness.

Proposition 4.1.2. *Let E, F, M be three vector spaces, and $\phi : E \times F \to M$ be a bilinear map. Then E and F are $\phi-$linearly disjoint if and only if:*

(LD') *For any $r, s \in \mathbb{N}$, $x_1, \dots, x_r$ linearly independent in E and $y_1, \dots, y_s$ linearly independent in F, the set $\{\phi(x_i, y_j) : i = 1, \dots, r,\ j = 1, \dots, s\}$ consists of linearly independent vectors in M.*

Proof.

($\Rightarrow$) Let $x_1, \dots, x_r$ be linearly independent in E and $y_1, \dots, y_s$ be linearly independent in F. Suppose that $\sum_{i=1}^r \sum_{j=1}^s \lambda_{ij}\phi(x_i, y_j) = 0$ for some $\lambda_{ij} \in \mathbb{K}$. Then, using the bilinearity of ϕ and setting $z_i := \sum_{j=1}^s \lambda_{ij} y_j$, we easily get $\sum_{i=1}^r \phi(x_i, z_i) = 0$. As the x_i's are linearly independent in E, we derive from (LD) that all z_i's have to be zero. This means that for each $i \in \{1, \dots, r\}$ we have $\sum_{j=1}^s \lambda_{ij} y_j = 0$, which implies by the linearly independence of the y_j's that $\lambda_{ij} = 0$ for all $i \in \{1, \dots, r\}$ and all $j \in \{1, \dots, s\}$.

($\Leftarrow$) Let $r \in \mathbb{N}$, $\{x_1, \dots, x_r\} \subseteq E$ and $\{y_1, \dots, y_r\} \subseteq F$ be such that $\sum_{i=1}^r \phi(x_i, y_i) = 0$. Suppose that the x_is are linearly independent and let $\{z_1, \dots, z_s\}$ be a basis of $span\{y_1, \dots, y_r\}$. Then for each $i \in \{1, \dots, r\}$ there exist $\lambda_{ij} \in \mathbb{K}$ s.t. $y_i = \sum_{j=1}^s \lambda_{ij} z_j$ and so by the bilinearity of ϕ we get:

$$0 = \sum_{i=1}^{r} \phi(x_i, y_j) = \sum_{i=1}^{r} \sum_{j=1}^{s} \lambda_{ij} \phi(x_i, z_j). \tag{4.1}$$

By applying (LD') to the x_i's and $z_j's$, we get that all $\phi(x_i, z_j)$'s are linearly independent. Therefore, (4.1) gives that $\lambda_{ij} = 0$ for all $i \in \{1, \dots, r\}$ and all $j \in \{1, \dots, s\}$ and so $y_i = 0$ for all $i \in \{1, \dots, r\}$. Exchanging the roles of the x_i's and the y_i's we get that (LD) holds. $\square$

Definition 4.1.3. *A tensor product of two vector spaces E and F over $\mathbb{K}$ is a pair (M, ϕ) consisting of a vector space M over $\mathbb{K}$ and of a bilinear map $\phi : E \times F \to M$ (canonical map) s.t. the following conditions are satisfied:*

(TP1) *The image of $E \times F$ spans the whole space M.*

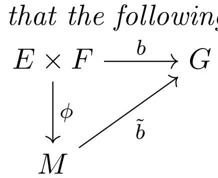
(TP2) *E and F are $\phi-$linearly disjoint.*

We now show that the tensor product of any two vector spaces always exists, satisfies the "universal property" and it is unique up to isomorphisms. For this reason, the tensor product of E and F is usually denoted by $E \otimes F$ and the canonical map by $(x, y) \mapsto x \otimes y$.

Theorem 4.1.4. *Let E, F be two vector spaces over $\mathbb{K}$.*

(a) There exists a tensor product of E and F.

(b) Let (M, ϕ) be a tensor product of E and F. Let G be any vector space over $\mathbb{K}$, and b any bilinear mapping of $E \times F$ into G. There exists a unique linear map $\tilde{b} : M \to G$ such that the following diagram is commutative.



(c) If (M_1, ϕ_1) and (M_2, ϕ_2) are two tensor products of E and F, then there is a bijective linear map u such that the following diagram is commutative.

$$\begin{array}{ccc} E \times F & \xrightarrow{\phi_2} & M_2 \\ \Big\downarrow{\scriptstyle \phi_1} & \nearrow{\scriptstyle u} & \\ M_1 & & \end{array}$$

Proof.

(a) Let $\mathcal{H}$ be the vector space of all functions from $E \times F$ into $\mathbb{K}$ which vanish outside a finite set ($\mathcal{H}$ is often called the free space of $E \times F$). For any $(x, y) \in E \times F$, let us define the function $e_{(x,y)} : E \times F \to \mathbb{K}$ as follows:

$$e_{(x,y)}(z, w) := \begin{cases} 1 & \text{if } (z, w) = (x, y) \\ 0 & \text{otherwise} \end{cases}.$$

Then $\mathcal{B}_{\mathcal{H}} := \{e_{(x,y)} : (x, y) \in E \times F\}$ forms a basis of $\mathcal{H}$ and so $\forall\, h \in \mathcal{H}$, $\exists!\ \lambda_{xy} \in \mathbb{K}$ s.t. $h = \sum_{x \in E} \sum_{y \in F} \lambda_{xy} e_{(x,y)}$ with $\lambda_{xy} = 0$ for all but finitely many x's in E and y's in Y. Let us consider now the following linear subspace of $\mathcal{H}$:

$$N := span\left\{ e_{\left(\sum\limits_{i=1}^{n} a_i x_i, \sum\limits_{j=1}^{m} b_j y_j\right)} - \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j e_{(x_i, y_j)} \; : \; n, m \in \mathbb{N}, a_i, b_j \in \mathbb{K}, (x_i, y_j) \in E \times F \right\}.$$

We then denote by M the quotient vector space $\mathcal{H}/N$, by π the quotient map from $\mathcal{H}$ onto M and by

$$\begin{array}{rccl} \phi : & E \times F & \to & M \\ & (x, y) & \to & \phi(x, y) := \pi\left(e_{(x,y)}\right). \end{array}$$

It is easy to see that the map ϕ is bilinear. Let us just show the linearity in the first variable as the proof in the second variable is just symmetric. Fixed $y \in F$, for any $a, b \in \mathbb{K}$ and any $x_1, x_2 \in E$, we get that:

$$\begin{aligned} \phi(ax_1 + bx_2, y) - a\phi(x_1, y) - b\phi(x_2, y) &= \pi\left(e_{(ax_1+bx_2,y)}\right) - a\pi\left(e_{(x_1,y)}\right) - b\pi\left(e_{x_2,y)}\right) \\ &= \pi\left(e_{(ax_1+bx_2,y)} - ae_{(x_1,y)} - be_{(x_2,y)}\right) \\ &= 0, \end{aligned}$$

where the last equality holds since $e_{(ax_1+bx_2,y)} - ae_{(x_1,y)} - be_{(x_2,y)} \in N$.

We aim to show that (M, ϕ) is a tensor product of E and F. It is clear from the definition of ϕ that

$$span(\phi(E \times F)) = span(\pi(\mathcal{B}_{\mathcal{H}})) = \pi(\mathcal{H}) = M,$$

i.e. (TP1) holds. It remains to prove that E and F are $\phi-$linearly disjoint. Let $r \in \mathbb{N}$, $\{x_1, \ldots, x_r\} \subseteq E$ and $\{y_1, \ldots, y_r\} \subseteq F$ be such that $\sum_{i=1}^{r} \phi(x_i, y_i) = 0$. Suppose that the y_i's are linearly independent. For any $\varphi \in E^*$, let us define the linear mapping $A_\varphi : \mathcal{H} \to F$ by setting $A_\varphi(e_{(x,y)}) := \varphi(x)y$. Then it is easy to check that A_φ vanishes on N, so it induces a map $\tilde{A}_\varphi : M \to F$ s.t. $\tilde{A}_\varphi(\pi(f)) = A(f)$, $\forall f \in \mathcal{H}$. Hence, since $\sum_{i=1}^{r} \phi(x_i, y_i) = 0$ can be rewritten as $\pi\left(\sum_{i=1}^{r} e_{(x_i,y_i)}\right) = 0$, we get that

$$0 = \tilde{A}_\varphi\left(\pi\left(\sum_{i=1}^{r} e_{(x_i,y_i)}\right)\right) = A_\varphi\left(\sum_{i=1}^{r} e_{(x_i,y_i)}\right) = \sum_{i=1}^{r} A_\varphi(e_{(x_i,y_i)}) = \sum_{i=1}^{r} \varphi(x_i)y_i.$$

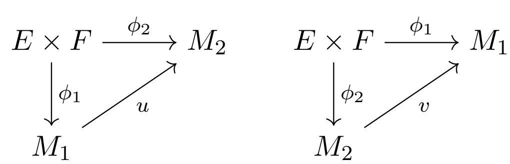
This together with the linear independence of the y_i's implies $\varphi(x_i) = 0$ for all $i \in \{1, \ldots, r\}$. Since the latter holds for all $\varphi \in E^*$, we have that $x_i = 0$ for all $i \in \{1, \ldots, r\}$. Exchanging the roles of the x_i's and the y_i's we get that (LD) holds, and so does (TP2) .

(b) Let (M, ϕ) be a tensor product of E and F, G a vector space and $b : E \times F \to G$ a bilinear map. Consider $\{x_\alpha\}_{\alpha \in A}$ and $\{y_\beta\}_{\beta \in B}$ bases of E and F, respectively. We know that $\{\phi(x_\alpha, y_\beta) : \alpha \in A, \beta \in B\}$ forms a basis of M, as *span*$(\phi(E \times F)) = M$ and, by Proposition 4.1.2, (LD') holds so the $\phi(x_\alpha, y_\beta)$'s for all $\alpha \in A$ and all $\beta \in B$ are linearly independent. The linear mapping $\tilde{b}$ will therefore be the unique linear map of M into G such that

$$\forall\, \alpha \in A, \forall\, \beta \in B,\ \tilde{b}(\phi(x_\alpha, y_\beta)) = b(x_\alpha, y_\beta).$$

Hence, the diagram in (b) commutes.

(c) Let (M_1, ϕ_1) and (M_2, ϕ_2) be two tensor products of E and F. Then using twice the universal property (b) we get that there exist unique linear maps $u : M_1 \to M_2$ and $v : M_2 \to M_1$ such that the following diagrams both commute:



Then combining $u \circ \phi_1 = \phi_2$ with $v \circ \phi_2 = \phi_1$, we get that u and v are one the inverse of the other. Hence, there is an algebraic isomorphism between M_1 and M_2. □

It is now natural to introduce the concept of tensor product of linear maps.

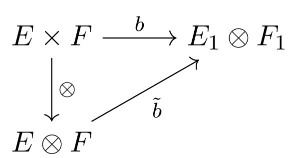
Proposition 4.1.5. *Let E, F, E_1, F_1 be four vector spaces over $\mathbb{K}$, and let $u : E \to E_1$ and $v : F \to F_1$ be linear mappings. There is a unique linear map of $E \otimes F$ into $E_1 \otimes F_1$, called the* tensor product of u and v *and denoted by $u \otimes v$, such that*

$$(u \otimes v)(x \otimes y) = u(x) \otimes v(y), \quad \forall\, x \in E, \forall\, y \in F.$$

Proof.
Let us define the mapping

$$\begin{array}{rccl} b: & E \times F & \to & E_1 \otimes F_1 \\ & (x, y) & \mapsto & b(x, y) := u(x) \otimes v(y), \end{array}$$

which is clearly bilinear because of the linearity of u and v and the bilinearity of the canonical map of the tensor product $E_1 \otimes F_1$. Then by the universal property there is a unique linear map $\tilde{b} : E \otimes F \to E_1 \otimes F_1$ s.t. the following diagram commutes:



i.e. $\tilde{b}(x \otimes y) = b(x, y)$, $\forall\,(x, y) \in E \times F$. Hence, using the definition of b, we get that $\tilde{b} \equiv u \otimes v$. □

Examples 4.1.6.

1. *Let $n, m \in \mathbb{N}$, $E = \mathbb{K}^n$ and $F = \mathbb{K}^m$. Then $E \otimes F = \mathbb{K}^{n \times m}$ is a tensor product of E and F whose canonical bilinear map ϕ is given by:*

$$\begin{array}{rccl} \phi: & E \times F & \to & \mathbb{K}^{n\times m} \\ & \Big((x_i)_{i=1}^n, (y_j)_{j=1}^m\Big) & \mapsto & (x_i y_j)_{1 \le i \le n, 1 \le j \le m}. \end{array}$$

2. *Let X and Y be two sets. For any functions $f : X \to \mathbb{K}$ and $g : Y \to \mathbb{K}$, we define:*

$$\begin{array}{rccl} f \otimes g: & X \times Y & \to & \mathbb{K} \\ & (x, y) & \mapsto & f(x)g(y). \end{array}$$

   Let E (resp. F) be the linear space of all functions from X (resp. Y) to $\mathbb{K}$ endowed with the usual addition and multiplication by scalars. We *denote by M the linear subspace of the space of all functions from $X \times Y$ to $\mathbb{K}$ spanned by the elements of the form $f \otimes g$ for all $f \in E$ and $g \in F$. Then M is actually a tensor product of E and F (see Exercise Sheet 7).*

Given X and Y open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, we can use the definitions in Example 2 above to construct the tensors $\mathcal{C}^k(X) \otimes \mathcal{C}^l(Y)$ for any $1 \leq k, l \leq \infty$. Then it is possible to show the following result (see e.g. [5, Theorem 39.2] for a proof).

Theorem 4.1.7. *Let X and Y open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Then $\mathcal{C}_c^\infty(X) \otimes \mathcal{C}_c^\infty(Y)$ is sequentially dense in $\mathcal{C}_c^\infty(X \times Y)$ endowed with the $\mathcal{C}^\infty-$topology.*

4.2 Topologies on the tensor product of locally convex t.v.s.

Given two locally convex t.v.s. E and F, there various ways to construct a topology on the tensor product $E \otimes F$ which makes the vector space $E \otimes F$ in a t.v.s.. Indeed, starting from the topologies on E and F, one can define a topology on $E \otimes F$ either relying directly on the seminorms on E and F, or using an embedding of $E \otimes F$ in some space related to E and F over which a natural topology already exists. The first method leads to the so-called $\pi-$topology. The second method may lead instead to a variety of topologies, the most important of which is the so-called $\varepsilon-$topology that is based on the isomorphism between $E \otimes F$ and $B(E'_\sigma, F'_\sigma)$ (see Proposition 4.2.9).

4.2.1 $\pi-$topology

Let us define the first main topology on $E \otimes F$ which we will see can be directly characterized by mean of the seminorms generating the topologies on the starting locally convex t.v.s. E and F.

Definition 4.2.1 ($\pi-$topology)**.**
Given two locally convex t.v.s. E and F, we define the $\pi-$topology *(or* projective topology*) on $E \otimes F$ to be the finest locally convex topology on this vector space for which the canonical mapping $E \times F \to E \otimes F$ is continuous. The space $E \otimes F$ equipped with the $\pi-$topology will be denoted by $E \otimes_\pi F$.*

A basis of neighbourhoods of the origin in $E \otimes_\pi F$ is given by the family:

$$\mathcal{B}_\pi := \{conv_b(U_\alpha \otimes V_\beta) : U_\alpha \in \mathcal{B}_E, V_\beta \in \mathcal{B}_F\},$$

where $\mathcal{B}_E$ (resp. $\mathcal{B}_F$) is a basis of neighbourhoods of the origin in E (resp. in F), $U_\alpha \otimes V_\beta := \{x \otimes y \in E \otimes F : x \in U_\alpha, y \in V_\beta\}$ and $conv_b(U_\alpha \otimes V_\beta)$ denotes the smallest convex balanced subset of $E \otimes F$ containing $U_\alpha \otimes V_\beta$. Indeed, by Theorem 4.1.14 in TVS-I, the topology generated by $\mathcal{B}_\pi$ is a locally convex topology $E \otimes F$ and it makes continuous the canonical map $\otimes$, since for any $U_\alpha \in \mathcal{B}_E$ and $V_\beta \in \mathcal{B}_F$ we have that $\otimes^{-1}(conv_b(U_\alpha \otimes V_\beta)) \supseteq \otimes^{-1}(U_\alpha \otimes V_\beta) = U_\alpha \times V_\beta$ which is a neighbourhood of the origin in $E \times F$. Hence, the topology generated by $\mathcal{B}_\pi$ is coarser than the $\pi-$topology. Moreover, the $\pi-$topology is by definition locally convex and so it has a basis $\mathcal{B}$ of convex balanced neighbourhoods of the origin in $E \otimes F$. Then, as the canonical mapping $\otimes$ is continuous w.r.t. the $\pi-$topology, we have that for any $C \in \mathcal{B}$ there exist $U_\alpha \in \mathcal{B}_E$ and $V_\beta \in \mathcal{B}_F$ s.t. $U_\alpha \times V_\beta \subseteq \otimes^{-1}(C)$. Hence, $U_\alpha \otimes V_\beta \subseteq C$ and so $conv_b(U_\alpha \otimes V_\beta) \subseteq conv_b(C) = C$, which yields that the topology generated by $\mathcal{B}_\pi$ is finer than the $\pi-$topology.

The $\pi-$topology on $E \otimes F$ can be described by means of the seminorms defining the locally convex topologies on E and F. Indeed, we have the following characterization of the $\pi-$topology.

Proposition 4.2.2. *Let E and F be two locally convex t.v.s. and let $\mathcal{P}$ (resp. $\mathcal{Q}$) be a family of seminorms generating the topology on E (resp. on F). The $\pi-$topology on $E \otimes F$ is generated by the family of seminorms*

$$\{p \otimes q : p \in \mathcal{P}, q \in \mathcal{Q}\},$$

where for any $p \in \mathcal{P}, q \in \mathcal{Q}, \theta \in E \otimes F$ we define:

$$(p \otimes q)(\theta) := \inf\{\rho > 0 : \theta \in \rho \; conv_b(U_p \otimes V_q)\}$$

with $U_p := \{x \in E : p(x) \leq 1\}$ and $V_q := \{y \in F : q(y) \leq 1\}$.

Proof. (Exercise Sheet 7) □

The seminorm $p \otimes q$ on $E \otimes F$ defined in the previous proposition is called *tensor product of the seminorms p and q* (or *projective cross seminorm*) and it can be represented in a more practical way that shows even more directly its relation to the seminorms defining the topologies on E and F.

Theorem 4.2.3.
Let E and F be two locally convex t.v.s. and let $\mathcal{P}$ (resp. $\mathcal{Q}$) be a family of seminorms generating the topology on E (resp. on F). Then for any $p \in \mathcal{P}$ and any $q \in \mathcal{Q}$ we have that the following hold.

a) *For all $\theta \in E \otimes F$,*

$$(p \otimes q)(\theta) = \inf\left\{\sum_{k=1}^{r} p(x_k)q(y_k) : \theta = \sum_{k=1}^{r} x_k \otimes y_k, \, , x_k \in E, y_k \in F, r \in \mathbb{N}\right\}.$$

b) *For all $x \in E$ and $y \in F$, $(p \otimes q)(x \otimes y) = p(x)q(y)$.*

Proof.
a) As above, we set $U_p := \{x \in E : p(x) \leq 1\}$, $V_q := \{y \in F : q(y) \leq 1\}$ and $W_{pq} := conv_b(U_p \otimes V_q)$. Let $\theta \in E \otimes F$ and $\rho > 0$ such that $\theta \in \rho W_{pq}$.

Let us preliminarily observe that the condition "$\theta \in \rho W_{pq}$ for some $\rho > 0$" is equivalent to:

$$\begin{aligned} &\theta = \sum_{k=1}^{N} t_k x_k \otimes y_k \quad \text{with } \; N \in \mathbb{N}, \; t_k \in \mathbb{K}, \; x_k \in E \text{ and } \; y_k \in F \text{ s.t.} \\ &\sum_{k=1}^{N} |t_k| \leq \rho, \; p(x_k) \leq 1, \; q(y_k) \leq 1, \; \forall k \in \{1, \dots, N\}. \end{aligned} \tag{4.2}$$

If we set $\xi_k := t_k x_k$ and $\eta_k := y_k$, then we can rewrite the condition (4.2) as

$$\theta = \sum_{k=1}^{N} \xi_k \otimes \eta_k \text{ with } \sum_{k=1}^{N} p(\xi_k) q(\eta_k) \leq \rho.$$

Then $\inf \left\{ \sum_{k=1}^{N} p(\xi_k) q(\eta_k) : \theta = \sum_{k=1}^{N} \xi_k \otimes \eta_k, , \xi_k \in E, \eta_k \in F, N \in \mathbb{N} \right\} \leq \rho$. Since this is true for any $\rho > 0$ s.t. $\theta \in \rho W_{pq}$, we get:

$$\inf \left\{ \sum_{i=1}^{r} p(x_i) q(y_i) : \theta = \sum_{i=1}^{r} x_i \otimes y_i, \, x_i \in E, y_i \in F, r \in \mathbb{N} \right\} \leq (p \otimes q)(\theta).$$

Conversely, let us consider an arbitrary representation of θ, i.e.

$$\theta = \sum_{k=1}^{N} \xi_k \otimes \eta_k \text{ with } \xi_k \in E, \; \eta_k \in F, \; N \in \mathbb{N}.$$

Let $\rho > 0$ s.t. $\sum_{k=1}^{N} p(\xi_k) q(\eta_k) \leq \rho$ and $\varepsilon > 0$. Define

- $I_1 := \{k \in \{1, \dots, N\} : p(\xi_k) q(\eta_k) \neq 0\}$
- $I_2 := \{k \in \{1, \dots, N\} : p(\xi_k) \neq 0 \text{ and } q(\eta_k) = 0\}$
- $I_3 := \{k \in \{1, \dots, N\} : p(\xi_k) = 0 \text{ and } q(\eta_k) \neq 0\}$
- $I_4 := \{k \in \{1, \dots, N\} : p(\xi_k) = 0 \text{ and } q(\eta_k) = 0\}$

and set

- $\forall k \in I_1$, $x_k := \frac{\xi_k}{p(\xi_k)}$, $y_k := \frac{\eta_k}{q(\eta_k)}$, $t_k := p(\xi_k) q(\eta_k)$
- $\forall k \in I_2$, $x_k := \frac{\xi_k}{p(\xi_k)}$, $y_k := \frac{N}{\varepsilon} p(\xi_k) \eta_k$, $t_k := \frac{\varepsilon}{N}$
- $\forall k \in I_3$, $x_k := \frac{N}{\varepsilon} q(\eta_k) \xi_k$, $y_k := \frac{\eta_k}{q(\eta_k)}$, $t_k := \frac{\varepsilon}{N}$
- $\forall k \in I_4$, $x_k := \frac{N}{\varepsilon} \xi_k$, $y_k := \eta_k$, $t_k := \frac{\varepsilon}{N}$

Then $\forall k \in \{1, \ldots, N\}$ we have that $p(x_k) \leq 1$ and $q(y_k) \leq 1$. Also we get:

$$\begin{aligned}\sum_{k=1}^{N} t_k x_k \otimes y_k &= \sum_{k\in I_1} p(\xi_k)q(\eta_k)\frac{\xi_k}{p(\xi_k)} \otimes \frac{\eta_k}{q(\eta_k)} + \sum_{k\in I_2} \frac{\varepsilon}{N}\frac{\xi_k}{p(\xi_k)} \otimes \frac{N}{\varepsilon} p(\xi_k)\eta_k \\ &+ \sum_{k\in I_3} \frac{\varepsilon}{N}\frac{N}{\varepsilon} q(\eta_k)\xi_k \otimes \frac{\eta_k}{q(\eta_k)} + \sum_{k\in I_4} \frac{\varepsilon}{N}\frac{N}{\varepsilon}\xi_k \otimes \eta_k \\ &= \sum_{k=1}^{N} \xi_k \otimes \eta_k = \theta\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{N} |t_k| &= \sum_{k\in I_1} p(\xi_k)q(\eta_k) + \sum_{k\in(I_2\cup I_3\cup I_4)} \frac{\varepsilon}{N} \\ &= \sum_{k\in I_1} p(\xi_k)q(\eta_k) + |I_2 \cup I_3 \cup I_4|\frac{\varepsilon}{N} \\ &\leq \sum_{k=1}^{N} p(\xi_k)q(\eta_k) + \varepsilon \leq \rho + \varepsilon.\end{aligned}$$

Hence, by (4.2) we get that $\theta \in (\rho + \varepsilon)W_{pq}$. As this holds for any $\varepsilon > 0$, we have $\theta \in \rho W_{pq}$. Therefore, we obtain that $(p \otimes q)(\theta) \leq \rho$ and in particular $(p \otimes q)(\theta) \leq \sum_{k=1}^{N} p(\xi_k)q(\eta_k)$. This yields that

$$(p \otimes q)(\theta) \leq \inf\left\{\sum_{k=1}^{N} p(\xi_k)q(\eta_k) : \theta = \sum_{k=1}^{N} \xi_k \otimes \eta_k, \ , \xi_k \in E, \eta_k \in F, N \in \mathbb{N}\right\}.$$

b) Let $x \in E$ and $y \in F$. By using a), we immediately get that

$$(p \otimes q)(x \otimes y) \leq p(x)q(y).$$

Conversely, consider $M := span\{x\}$ and define $L : M \to \mathbb{K}$ as $L(\lambda x) := \lambda p(x)$ for all $\lambda \in \mathbb{K}$. Then clearly L is a linear functional on M and for any $m \in M$, i.e. $m = \lambda x$ for some $\lambda \in \mathbb{K}$, we have $|L(m)| = |\lambda|p(x) = p(\lambda x) = p(m)$. Therefore, Hahn-Banach theorem can be applied and provides that:

$$\exists\, x' \in E' \text{ s.t. } \langle x', x\rangle = p(x) \text{ and } |\langle x', x_1\rangle| \leq p(x_1), \forall\, x_1 \in E. \tag{4.3}$$

Repeating this reasoning for y we get that:

$$\exists\, y' \in F' \text{ s.t. } \langle y', y\rangle = q(y) \text{ and } |\langle y', y_1\rangle| \leq q(y_1), \forall\, y_1 \in F. \tag{4.4}$$

Let us consider now any representation of $x \otimes y$, namely $x \otimes y = \sum_{k=1}^{N} x_k \otimes y_k$ with $x_k \in E$, $y_k \in F$ and $N \in \mathbb{N}$. Then, combining Proposition 4.1.5 and the second part of both (4.3) and (4.4), we obtain:

$$\begin{aligned} \left|\langle x' \otimes y', x \otimes y\rangle\right| &\leq \sum_{k=1}^{N} |\langle x' \otimes y', x_k \otimes y_k\rangle| \\ &\overset{\text{Prop 4.1.5}}{=} \sum_{k=1}^{N} |\langle x', x_k\rangle| \cdot |\langle y', y_k\rangle| \\ &\overset{(4.3)\text{ and }(4.4)}{\leq} \sum_{k=1}^{N} p(x_k)q(x_k). \end{aligned}$$

Since this is true for any representation of $x \otimes y$, we deduce by a) that:

$$\left|\langle x' \otimes y', x \otimes y\rangle\right| \leq (p \otimes q)(x \otimes y).$$

The latter together with the first part of (4.3) and (4.4) gives:

$$p(x)q(y) = |p(x)| \cdot |q(y)| = |\langle x', x\rangle| \cdot |\langle y', y\rangle| = \left|\langle x' \otimes y', x \otimes y\rangle\right| \leq (p \otimes q)(x \otimes y).$$

□

Proposition 4.2.4. *Let E and F be two locally convex t.v.s.. $E \otimes_\pi F$ is Hausdorff if and only if E and F are both Hausdorff.*

Proof. (Exercise Sheet 7) □

Corollary 4.2.5. *Let (E, p) and (F, q) be seminormed spaces. Then $p \otimes q$ is a norm on $E \otimes F$ if and only if p and q are both norms.*

Proof.
Under our assumptions, the $\pi-$topology on $E \otimes F$ is generated by the single seminorm $p \otimes q$. Then, recalling that a seminormed space is normed iff it is Hausdorff and using Proposition 4.2.4, we get: $(E \otimes F, p \otimes q)$ is normed $\Leftrightarrow$ $E \otimes_\pi F$ is Hausdorff $\Leftrightarrow$ E and F are both Hausdorff $\Leftrightarrow$ (E, p) and (F, q) are both normed. □

Definition 4.2.6. *Let (E, p) and (F, q) be normed spaces. The normed space $(E \otimes F, p \otimes q)$ is called the* projective tensor product *of E and F and $p \otimes q$ is said to be the corresponding* projective tensor norm.

In analogy with the algebraic case (see Theorem 4.1.4-b), we also have a universal property for the space $E \otimes_\pi F$.

Proposition 4.2.7.
Let E, F be locally convex spaces. The $\pi-$topology on $E \otimes_\pi F$ is the unique locally convex topology on $E \otimes F$ such that the following property holds:

(UP) *For every locally convex space G, the algebraic isomorphism between the space of bilinear mappings from $E \times F$ into G and the space of all linear mappings from $E \otimes F$ into G (given by Theorem 4.1.4-b) induces an algebraic isomorphism between $B(E, F; G)$ and $L(E \otimes F; G)$, where $B(E, F; G)$ denotes the space of all continuous bilinear mappings from $E \times F$ into G and $L(E \otimes F; G)$ the space of all continuous linear mappings from $E \otimes F$ into G.*

Proof. We first show that the $\pi-$topology fulfills (UP). Let (G, ω) be a locally convex space and $b \in B(E, F; G)$, then Theorem 4.1.4-b) ensures that there exists a unique $\tilde{b} : E \otimes F \to G$ linear s.t. $\tilde{b} \circ \phi = b$, where $\phi : E \times F \to E \otimes F$ is the canonical mapping. Let U basic neighbourhood of the origin in G, so w.l.o.g. we can assume U convex and balanced. Then the continuity of b implies that there exist V basic neighbourhood of the origin in E and W basic neighbourhood of the origin in E s.t. $\tilde{b}(\phi(V \times W)) = b(V \times W) \subseteq U$. Hence, $\phi(V \times W) \subseteq \tilde{b}^{-1}(U)$ and so $conv_b(\phi(V \times W)) \subseteq conv_b(\tilde{b}^{-1}(U)) = \tilde{b}^{-1}(U)$, which shows the continuity of $\tilde{b} : E \otimes_\pi F \to (G, \omega)$ as $conv_b(\phi(V \times W)) \in \mathcal{B}_\pi$.

Let τ be a locally convex topology on $E \otimes F$ such that the property (UP) holds. Then (UP) holds in particular for $G = (E \otimes F, \tau)$. Therefore, since in the algebraic isomorphism given by Theorem 4.1.4-b) in this case the canonical mapping $\phi : E \times F \to E \otimes F$ corresponds to the identity $id : E \otimes F \to E \otimes F$, we get that $\phi : E \times F \to E \otimes_\tau F$ has to be continuous.

$$
\begin{array}{ccc}
E \times F & \xrightarrow{\phi} & E \otimes_\tau F \\
\downarrow \phi & \nearrow_{id} & \\
E \otimes_\tau F & &
\end{array}
$$

This implies that τ is coarser than the $\pi-$topology. On the other hand, (UP) also holds for $G = (E \otimes F, \pi)$. Hence,

$$
\begin{array}{ccc}
E \times F & \xrightarrow{\phi} & E \otimes_\pi F \\
\downarrow \phi & \nearrow_{id} & \\
E \otimes_\tau F & &
\end{array}
$$

since by definition of $\pi-$topology $\phi : E \times F \to E \otimes_\pi F$ is continuous, the $id : E \otimes_\tau F \to E \otimes_\pi F$ has to be also continuous. This means that the $\pi-$topology is coarser than τ, which completes the proof. □

Corollary 4.2.8. *$(E \otimes_\pi F)' \cong B(E,F)$, where $B(E,F) := B(E,F;\mathbb{K})$.*

Proof. By taking $G = \mathbb{K}$ in Proposition 4.2.7, we get the conclusion. □

4.2.2 $\varepsilon-$topology

The definition of $\varepsilon-$topology strongly relies strongly relies on the algebraic isomorphism between $E \otimes F$ and the space $B(E'_\sigma, F'_\sigma)$ of continuous bilinear forms on the product $E'_\sigma \times F'_\sigma$ of the weak duals of E and F (see Section 3.2 for the definition of weak topology). More precisely, the following hold.

Proposition 4.2.9. *Let E and F be non-trivial locally convex t.v.s. over $\mathbb{K}$ with non-trivial topological duals. The space $B(E'_\sigma, F'_\sigma)$ is a tensor product of E and F.*

Proof.
Let us consider the bilinear mapping:

$$\begin{array}{rccclcl} \phi: & E\times F & \to & B(E'_\sigma, F'_\sigma) & & & \\ & (x,y) & \mapsto & \phi(x,y): & E'_\sigma \times F'_\sigma & \to & \mathbb{K} \\ & & & & (x',y') & \mapsto & \langle x', x\rangle\langle y', y\rangle. \end{array} \tag{4.5}$$

We first show that E and F are ϕ-linearly disjoint. Let $r, s \in \mathbb{N}$, $x_1, \ldots, x_r$ be linearly independent in E and $y_1, \ldots, y_s$ be linearly independent in F. In their correspondence, select[1] $x'_1, \ldots, x'_r \in E'$ and $y'_1, \ldots, y'_s \in F'$ such that

$$\langle x'_m, x_j\rangle = \delta_{mj},\ \forall\, m, j \in \{1, \ldots, r\} \ \text{ and } \ \langle y'_n, y_k\rangle = \delta_{nk}\, \forall\, n, k \in \{1, \ldots, s\}.$$

Then we have that:

$$\phi(x_j, y_k)(x'_m, y'_n) = \langle x'_m, x_j\rangle\langle y'_n, y_k\rangle = \begin{cases} 1 & \text{if } m = j \text{ and } n = k \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

This implies that the set $\{\phi(x_j, y_k) : j = 1, \ldots, r,\ k = 1, \ldots, s\}$ consists of linearly independent elements. Indeed, if there exists $\lambda_{jk} \in \mathbb{K}$ s.t.

$$\sum_{j=1}^{r}\sum_{k=1}^{s} \lambda_{jk}\phi(x_j, y_k) = 0$$

then for all $m \in \{1, \ldots, r\}$ and all $n \in \{1, \ldots, r\}$ we have that:

$$\sum_{j=1}^{r}\sum_{k=1}^{s} \lambda_{jk}\phi(x_j, y_k)(x'_m, y'_n) = 0$$

[1] This can be done using Lemma 3.2.10 together with the assumption that E' and F' are not trivial.

and so by using (4.6) that all $\lambda_{mn} = 0$.

We have therefore showed that (LD') holds and so, by Proposition 4.1.2, E and F are ϕ-linearly disjoint. Let us briefly sketch the main steps of the proof that $span(\phi(E \times F)) = B(E'_\sigma, F'_\sigma)$.

a) Take any $\varphi \in B(E'_\sigma, F'_\sigma)$. By the continuity of φ, it follows that there exist finite subsets $A \subset E$ and $B \subset F$ s.t. $|\varphi(x', y')| \leq 1$, $\forall\, x' \in A^\circ$, $\forall\, y' \in B^\circ$.

b) Set $E_A := span(A)$ and $F_B := span(B)$. Since E_A and E_B are finite dimensional, their orthogonals $(E_A)^\circ$ and $(F_B)^\circ$ have finite codimension and so

$$E' \times F' = (M' \oplus (E_A)^\circ) \times (N' \oplus (F_B)^\circ) = (M' \times N') \oplus ((E_A)^\circ \times F') \oplus (E' \times (F_B)^\circ),$$

where M' and N' finite dimensional subspaces of E' and F', respectively.

c) Using a) and b) one can prove that φ vanishes on the direct sum $((E_A)^\circ \times F') \oplus (E' \times (F_B)^\circ)$ and so that φ is completely determined by its restriction to a finite dimensional subspace $M' \times N'$ of $E' \times F'$.

d) Let $r := dim(E_A)$ and $s := dim(F_B)$. Then there exist $x_1, \ldots, x_r \in E_A$ and $y_1, \ldots, y_s \in F_B$ s.t. the restriction of φ to $M' \times N'$ is given by

$$(x', y') \mapsto \sum_{i=1}^{r} \sum_{j=1}^{s} \langle x', x_i \rangle \langle y', y_j \rangle.$$

Hence, by c), we can conclude that $\phi \in span(\phi(E \times F))$.

□

The $\varepsilon-$topology on $E \otimes F$ will be then naturally defined by the so-called *topology of bi-equicontinuous convergence* on the space $B(E'_\sigma, F'_\sigma)$. As the name suggests this topology is intimately related to the notion *equicontinuous sets of linear mappings* between t.v.s..

Definition 4.2.10. *Let X and Y be two t.v.s.. A set S of linear mappings of X into Y is said to be* equicontinuous *if for any neighbourhood V of the origin in Y there exists a neighbourhood U of the origin in X such that*

$$\forall\, f \in S,\, x \in U \;\Rightarrow\; f(x) \in V$$

i.e.

$$\forall\, f \in S,\, f(U) \subseteq V \quad (or\ U \subseteq f^{-1}(V)).$$

The equicontinuity condition can be also rewritten as follows: S is equicontinuous if for any neighbourhood V of the origin in Y there exists a neighbourhood U of the origin in X such that $\bigcup_{f \in S} f(U) \subseteq V$ or, equivalently, if for any neighbourhood V of the origin in Y the set $\bigcap_{f \in S} f^{-1}(V)$ is a neighbourhood of the origin in X.

Note that if S is equicontinuous then each mapping $f \in S$ is continuous but clearly the converse does not hold.

A first property of equicontinuous sets which is clear from the definition is that any subset of an equicontinuous set is itself equicontinuous. We are going to introduce now few more properties of equicontinuous sets of linear functionals on a t.v.s. which will be useful in the following.

Proposition 4.2.11. *A set of continuous linear functionals on a t.v.s. X is equicontinuous if and only if it is contained in the polar of some neighbourhood of the origin in X.*

Proof.
For any $\rho > 0$, let us denote by $D_\rho := \{k \in \mathbb{K} : |k| \leq \rho\}$. Let H be an equicontinuous set of linear forms on X. Then there exists a neighbourhood U of the origin in X s.t. $\bigcup_{f\in H} f(U) \subseteq D_1$, i.e. $\forall f \in H$, $|\langle f, x\rangle| \leq 1, \forall\, x \in U$, which means exactly that $H \subseteq U^\circ$.

Conversely, let U be an arbitrary neighbourhood of the origin in X and let us consider the polar $U^\circ := \{f \in X' : \sup_{x\in U} |\langle f, x\rangle| \leq 1\}$. Then for any $\rho > 0$

$$\forall\, f \in U^\circ,\ |\langle f, y\rangle| \leq \rho,\ \forall\, y \in \rho U,$$

which is equivalent to

$$\bigcup_{f\in U^\circ} f(\rho U) \subseteq D_\rho.$$

This means that U° is equicontinuous and so any subset H of U° is also equicontinuous, which yields the conclusion. □

Proposition 4.2.12. *Let X be a non-trivial locally convex Hausdorff t.v.s.*[2]*. Any equicontinuous subset of X' is bounded in X'_σ.*

Proof. Let H be an equicontinuous subset of X'. Then, by Proposition 4.2.11, we get that there exists a neighbourhood U of the origin in X such that $H \subseteq U^\circ$. By Banach-Alaoglu theorem (see Theorem 3.3.3), we know that $U^\circ$is compact in X'_σ and so bounded by Proposition 2.2.4. Hence, by Proposition 2.2.2-4, H is also bounded in X'_σ. □

It is also possible to show, but we are not going to prove this here, that the following holds.

Proposition 4.2.13. *Let X be a non-trivial locally convex Hausdorff t.v.s.. The union of all equicontinuous subsets of X' is dense in X'_σ.*

[2] Recall that non-trivial locally convex Hausdorff t.v.s. have non-trivial topological dual by Proposition 3.2.8

Now let us come back to the space $B(X,Y;Z)$ of continuous bilinear mappings from $X \times Y$ to Z, where X, Y and Z are non-trivial locally convex t.v.s.. The following is a very natural way of introducing a topology on $B(X,Y;Z)$ and is a kind of generalization of the method we have used to define polar topologies in Chapter 3.

Consider a family Σ (resp. Γ) of bounded subsets of X (resp. Y) satisfying the following properties:
(P1) If $A_1, A_2 \in \Sigma$, then $\exists\, A_3 \in \Sigma$ s.t. $A_1 \cup A_2 \subseteq A_3$.
(P2) If $A_1 \in \Sigma$ and $\lambda \in \mathbb{K}$, then $\exists\, A_2 \in \Sigma$ s.t. $\lambda A_1 \subseteq A_2$.
(resp. satisfying (P1) and (P2) replacing Σ by Γ). The Σ-Γ-topology on $B(X,Y;Z)$, or topology of uniform convergence on subsets of the form $A \times B$ with $A \in \Sigma$ and $B \in \Gamma$, is defined by taking as a basis of neighbourhoods of the origin in $B(X,Y;Z)$ the following family:

$$\mathcal{U} := \{\mathcal{U}(A,B;W) :\ A \in \Sigma, B \in \Gamma, W \in \mathcal{B}_Z(o)\},$$

where

$$\mathcal{U}(A,B;W) := \{\varphi \in B(X,Y;Z) : \varphi(A,B) \subseteq W\}$$

and $\mathcal{B}_Z(o)$ is a basis of neighbourhoods of the origin in Z. It is not difficult to verify that (c.f. [5, Chapter 32]):

a) each $\mathcal{U}(A,B;W)$ is an absorbing, convex, balanced subset of $B(X,Y;Z)$;
b) the Σ-Γ-topology makes $B(X,Y;Z)$ into a locally convex t.v.s. (by Theorem 4.1.14 of TVS-I);
c) If Z is Hausdorff, the union of all subsets in Σ is dense in X and the union of all subsets in Γ is dense in Y, then the Σ-Γ-topology on $B(X,Y;Z)$ is Hausdorff.

In particular, given two non-trivial locally convex Hausdorff t.v.s. E and F, we call *topology of bi-equicontinuous convergence* on $B(E'_\sigma, F'_\sigma)$ the Σ-Γ-topology when Σ is the family of all equicontinuous subsets of E' and Γ is the family of all equicontinuous subsets of F'. Note that we can make this choice of Σ and Γ, because by Proposition 4.2.12 all equicontinuous subsets of E' (resp. F') are bounded in E'_σ (resp. F'_σ) and satisfy the properties $(P1)$ and $(P2)$. A basis for the topology of bi-equicontinuous convergence $B(E'_\sigma, F'_\sigma)$ is then given by:

$$\mathcal{U} := \{\mathcal{U}(A,B;\varepsilon) :\ A \in \Sigma, B \in \Gamma, \varepsilon > 0\}$$

where

$$\begin{aligned} \mathcal{U}(A,B;\varepsilon) &:= \{\varphi \in B(E'_\sigma, F'_\sigma) : \varphi(A,B) \subseteq D_\varepsilon\} \\ &= \{\varphi \in B(E'_\sigma, F'_\sigma) : |\varphi(x',y')| \leq \varepsilon, \forall x' \in A, \forall y' \in B\} \end{aligned}$$

and $D_\varepsilon := \{k \in \mathbb{K} : |k| \leq \varepsilon\}$. By using a) and b), we get that $B(E'_\sigma, F'_\sigma)$ endowed with the topology of bi-equicontinuous convergence is a locally convex t.v.s.. Also, by using Proposition 4.2.13 together with c), we can prove that the topology of bi-equicontinuous convergence on $B(E'_\sigma, F'_\sigma)$ is Hausdorff (as E and F are both assumed to be Hausdorff).

We can then use the isomorphism between $E \otimes F$ and $B(E'_\sigma, F'_\sigma)$ provided by Proposition 4.2.9 to carry the topology of bi-equicontinuous convergence on $B(E'_\sigma, F'_\sigma)$ over $E \otimes F$.

Definition 4.2.14 ($\varepsilon-$topology)**.**
*Given two non-trivial locally convex Hausdorff t.v.s. E and F, we define the $\varepsilon-$*topology *on $E \otimes F$ to be the topology carried over from $B(E'_\sigma, F'_\sigma)$ endowed with the topology of bi-equicontinuous convergence, i.e. topology of uniform convergence on the products of an equicontinuous subset of E' and an equicontinuous subset of F'. The space $E \otimes F$ equipped with the $\varepsilon-$topology will be denoted by $E \otimes_\varepsilon F$.*

It is clear then $E \otimes_\varepsilon F$ is a locally convex Hausdorff t.v.s.. Moreover, we have that:

Proposition 4.2.15. *Given two non-trivial locally convex Hausdorff t.v.s. E and F, the canonical mapping from $E \times F$ into $E \otimes_\varepsilon F$ is continuous. Hence, the $\pi-$topology is finer than the $\varepsilon-$topology on $E \otimes F$.*

Proof.
By definition of $\pi-$topology and $\varepsilon-$topology, it is enough to show that the canonical mapping ϕ from $E \times F$ into $B(E'_\sigma, F'_\sigma)$ defined in (4.5) is continuous w.r.t. the topology of bi-equicontinuous convergence on $B(E'_\sigma, F'_\sigma)$. Let $\varepsilon > 0$, A any equicontinuous subset of E' and B any equicontinuous subset of F', then by Proposition 4.2.11 we get that there exist a neighbourhood N_A of the origin in E and a neighbourhood N_B of the origin in F s.t. $A \subseteq (N_A)^\circ$ and $B \subseteq (N_B)^\circ$. Hence, we obtain that

$$\begin{aligned}
\phi^{-1}(\mathcal{U}(A,B;\varepsilon)) &= \{(x,y) \in E \times F : \phi(x,y) \in \mathcal{U}(A,B;\varepsilon)\} \\
&= \left\{(x,y) \in E \times F : |\phi(x,y)(x',y')| \leq \varepsilon, \forall\, x' \in A, \forall\, y' \in B\right\} \\
&= \left\{(x,y) \in E \times F : |\langle x', x\rangle\langle y', y\rangle| \leq \varepsilon, \forall\, x' \in A, \forall\, y' \in B\right\} \\
&\supseteq \left\{(x,y) \in E \times F : |\langle x', x\rangle\langle y', y\rangle| \leq \varepsilon, \forall\, x' \in (N_A)^\circ, \forall\, y' \in (N_B)^\circ\right\} \\
&\supseteq \varepsilon N_A \times N_B,
\end{aligned}$$

which proves the continuity of ϕ as $\varepsilon N_A \times N_B$ is a neighbourhood of the origin in $E \times F$. □

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