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TOPOLOGICAL VECTOR SPACES II–WS 2019/2020

Bonus Sheet - Solution

1) Given two sets X and Y, let E (resp. F) be the linear space of all functions from X (resp. Y) to $\mathbb{K}$ endowed with the usual addition and multiplication by scalars. For any $f \in E$ and $g \in F$, define:

$$\begin{array}{rccc} f \otimes g\colon & X \times Y & \to & \mathbb{K} \\ & (x,y) & \mapsto & f(x)g(y). \end{array}$$

Show that $M := \text{span}\{f \otimes g : f \in E, g \in F\}$ is a tensor product of E and F.

Proof. Define the map $\phi\colon E \times F \to M$ as $\phi(f,g) := f \otimes g$. Since $M = \text{span}\{\phi(E \times F)\}$, to prove that (M,ϕ) is a tensor product of E and F we only need to show that ϕ is bilinear and that E and F are ϕ–linearly disjoint.

Let $\lambda \in \mathbb{K}, f, g \in E$ and $h \in F$. Then, for all $(x,y) \in X \times Y$ we have

$$\begin{aligned} ((f+\lambda g) \otimes h)(x,y) &= (f+\lambda g)(x)h(y) \\ &= f(x)h(y) + \lambda g(x)h(y) \\ &= (f \otimes h)(x,y) + \lambda (g \otimes h)(x,y), \end{aligned}$$

i.e. $\phi(f+\lambda g, h) = \phi(f,h) + \lambda\phi(g,h)$. This proves the linearity of ϕ in its first argument. The linearity in the second argument can be analogously proved.

Let $r \in \mathbb{N}$ and $\{f_1, \dots, f_r\} \subseteq E$ and $\{g_1, \dots, g_r\} \subseteq F$ be such that $\sum_{i=1}^{r} \phi(f_i, g_i) = 0$, i.e.

$$0 = \sum_{i=1}^{r} (f_i \otimes g_i)(x,y) = \sum_{i=1}^{r} f_i(x) g_i(y) \text{ for all } x \in X, y \in Y. \tag{1}$$

If $\{f_1, \dots, f_r\}$ is linearly independent, then (1) yields $g_1(y) = \cdots = g_r(y) = 0$ for all $y \in Y$. Thus, $g_1 = \cdots = g_r = 0$ on Y. If $\{g_1, \dots, g_r\}$ is linearly independent, then (1) yields $f_1 = \cdots = f_r = 0$ on X.

Hence, E and F are ϕ–linearly disjoint. □

Let E and F be two locally convex t.v.s. over $\mathbb{K}$. Denote by $E \otimes_\pi F$ the tensor product $E \otimes F$ endowed with the π–topology. Prove the following statements:

2) If $\mathcal{P}$ (resp. $\mathcal{Q}$) is a family of seminorms generating the topology on E (resp. on F), then the π–topology on $E \otimes F$ is generated by the family

$$\{p \otimes q : p \in \mathcal{P}, q \in \mathcal{Q}\},$$

where for any $p \in \mathcal{P}, q \in \mathcal{Q}, \theta \in E \otimes F$ we define:

$$(p \otimes q)(\theta) := \inf\{\rho > 0 : \theta \in \rho \operatorname{conv}_b(U_p \otimes V_q)\}$$

with $U_p := \{x \in E : p(x) \leq 1\}$ and $U_q := \{y \in F : q(y) \leq 1\}$.

Proof. W.l.o.g. we may assume that the families $\mathcal{P}$ and $\mathcal{Q}$ are directed. Therefore, $\mathcal{B}_\mathcal{P} = \{\varepsilon U_p : p \in \mathcal{P}, \varepsilon > 0\}$ resp. $\mathcal{B}_\mathcal{Q} = \{\varepsilon U_q : q \in \mathcal{Q}, \varepsilon > 0\}$ are bases of neighbourhoods of the origin in E resp. in F (see TVS–I, Section 4.2). Then

$$\mathcal{B}_\pi := \{\operatorname{conv}_b(\delta U_p \otimes \varepsilon U_q) : p \in \mathcal{P}, q \in \mathcal{Q}, \delta > 0, \varepsilon > 0\}$$

is a basis of neighbourhoods of the origin in the π–topology on $E \otimes F$ (see Section 4.2.1). Since $\mathcal{P}$ and $\mathcal{Q}$ are directed, the family $\mathcal{P} \otimes \mathcal{Q} := \{p \otimes q : p \in \mathcal{P}, q \in \mathcal{Q}\}$ is also directed, because

$$\max\{p_1 \otimes q_1, p_2 \otimes q_2\} \leq (\max\{p_1, p_2\}) \otimes (\max\{q_1, q_2\})$$

holds for all $p_1, p_2 \in \mathcal{P}, q_1, q_2 \in \mathcal{Q}$. Thus,

$$\mathcal{B}_{\mathcal{P}\otimes\mathcal{Q}} := \{\lambda U_{p\otimes q} : p \in \mathcal{P}, q \in \mathcal{Q}, \lambda > 0\}$$

is a basis of neighbourhoods of the origin on $E \otimes F$ endowed with the topology generated by $\mathcal{P} \otimes \mathcal{Q}$. Hence, it suffices to show that $\operatorname{conv}_b(\delta U_p \otimes \varepsilon U_q) = (\delta\varepsilon) U_{p\otimes q}$ for all $p \in \mathcal{P}, q \in \mathcal{Q}$ and all $\delta > 0, \varepsilon > 0$ as this implies that $\mathcal{B}_\pi = \mathcal{B}_{\mathcal{P}\otimes\mathcal{Q}}$.

Let us first show that for any seminorms p on E and q on F we have

$$U_{p\otimes q} = \operatorname{conv}_b(U_p \otimes U_q). \tag{2}$$

Set $W_{pq} := \operatorname{conv}_b(U_p \otimes U_q)$. If $\theta \in U_{p\otimes q}$, then $(p \otimes q)(\theta) \leq 1$, i.e. for any $\varepsilon > 0$ there is $\rho > 0$ such that

$$\theta \in \rho W_{pq} \quad \text{and} \quad \rho < (p \otimes q)(\theta) + \varepsilon \leq 1 + \varepsilon.$$

Since W_{pq} is balanced by definition, we have that $\theta \in \rho W_{pq} \subseteq (1+\varepsilon) W_{pq}$. Hence, by the arbitrarity of ε, we get $\theta \in W_{pq}$. Conversely, if $\theta \in W_{pq} = 1 \cdot W_{pq}$, then $(p \otimes q)(\theta) \leq 1$ and so $\theta \in U_{p\otimes q}$.

Now, for any $p \in \mathcal{P}, q \in Q, \delta > 0, \varepsilon > 0$ we have that $\delta^{-1}p$ and $\varepsilon^{-1}q$ are seminorms. Thus,

$$\operatorname{conv}_b(\delta U_p \otimes \varepsilon U_q) = \operatorname{conv}_b(U_{\delta^{-1}p} \otimes U_{\varepsilon^{-1}q}) \overset{(2)}{=} U_{(\delta^{-1}p)\otimes(\varepsilon^{-1}q)} = U_{(\delta\varepsilon)^{-1}p\otimes q} = (\delta\varepsilon) U_{p\otimes q},$$

which proves the claim. □

3) $E \otimes_\pi F$ is Hausdorff if and only if E and F are both Hausdorff.

Proof. Let $\mathcal{P}$ (resp. $\mathcal{Q}$) be a family of seminorms generating the topology on E (resp. F) and let $\mathcal{P} \otimes \mathcal{Q}$ be defined as in Exercise 2). Then, by Proposition 4.3.3 from TVS–I, the spaces E, F and $E \otimes_\pi F$ are Hausdorff if and only if $\mathcal{P}, \mathcal{Q}$ and $\mathcal{P} \otimes \mathcal{Q}$ are separating.

Assume that $E \otimes_\pi F$ is Hausdorff and so that $\mathcal{P} \otimes \mathcal{Q}$ is separating. Let $x \in E \setminus \{0\}, y \in F \setminus \{0\}$. Then $x \otimes y \neq 0 \in E \otimes F$ and, hence, there are $p \in \mathcal{P}$ and $q \in \mathcal{Q}$ such that

$$0 \neq (p \otimes q)(x \otimes y) = p(x)q(y),$$

where the last equality is due to Theorem 4.2.3. Thus, $p(x) \neq 0$ and $q(y) \neq 0$, which imply that the families $\mathcal{P}$ and $\mathcal{Q}$ are both separating and so E and F are both Hausdorff.

Conversely, assume that E and F are Hausdorff. Let $0 \neq \theta \in E \otimes F$, say

$$\theta = \sum_{k=1}^{r} x_k \otimes y_k,$$

where $x_1, \ldots, x_r \in E$ (resp. $y_1, \ldots, y_r \in F$) can be assumed to be linearly independent. Since E and F are Hausdorff, by Lemma 3.2.10 (a consequence of the Hahn-Banach Theorem), there are $x' \in E'$ and $y' \in F'$ such that

$$\langle x', x_1 \rangle = \langle y', y_1 \rangle = 1 \quad \text{and} \quad \langle x', x_k \rangle = \langle y', y_k \rangle = 0 \text{ for } k \geq 2.$$

Then the linear map

$$\begin{array}{rlcl} \theta' : & E \otimes F & \to & \mathbb{K} \\ & \sum_{l=1}^{s} \xi_l \otimes \eta_l & \mapsto & \sum_{l=1}^{r} \langle x', \xi_l \rangle \langle y', \eta_l \rangle \end{array}$$

is continuous w.r.t. π–topology and satisfies $\langle \theta', \theta \rangle = 1$ by construction. In particular, θ' is $(p \otimes q)$–continuous for some $p \in \mathcal{P}$ and $q \in \mathcal{Q}$, i.e. there is some $C > 0$ such that

$$0 < 1 = \left| \langle \theta', \theta \rangle \right| \leq C(p \otimes q)(\theta).$$

This yields $(p \otimes q)(\theta) \neq 0$. Thus, the family $\mathcal{P} \otimes \mathcal{Q}$ is separating and the claim follows. □