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TOPOLOGICAL VECTOR SPACES II–WS 2017/2018

Christmas Assignment

*This exercise sheet aims to assess your progress and to explicitly work out more details of some of the results proposed in the previous lectures. Please, hand in your solutions in postbox 18 near F411 by Wednesday the 8th of January at 13:30. The solutions to this assignment will be discussed in the tutorial on **Thursday** the **16th** of **January** (**13:30**–15:00) in **D406**.*

1) Show that any linear map f from an LF-space E into a Fréchet space F whose graph is closed is continuous.

2) Use the closed graph theorem to prove Proposition 1 in Exercise Sheet 2.

Proposition 1. *Let τ_1 and τ_2 be two topologies on a vector space X such that (X, τ_i) is a Fréchet space for $i = 1, 2$. If $\tau_1 \cap \tau_2$ is a Hausdorff topology on X, then $\tau_1 = \tau_2$.*

3) Let $l_1 := \{x = (x_n)_{n\in\mathbb{N}} \subset \mathbb{R}, \|x\|_1 < \infty\}$ and $l_\infty := \{x = (x_n)_{n\in\mathbb{N}} \subset \mathbb{R}, \|x\|_\infty < \infty\}$, where $\|x\|_1 := \sum_{n=1}^{\infty} |x_n|$ and $\|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n|$ for all $x = (x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$. Since l_1 and l_∞ have an vector space basis of the same cardinality, there exists a vector space isomorphism $\varphi\colon l_1 \to l_\infty$ and $\|x\|' := \|\varphi(x)\|_\infty$ for all $x \in l_1$ defines a norm on l_1.

Show that both $(l_1, \|\cdot\|_1)$ and $(l_1, \|\cdot\|')$ are Fréchet spaces but the topologies τ_1 and τ' induced by $\|\cdot\|_1$ and $\|\cdot\|'$, respectively, are not comparable.

4) Let X be a Hausdorff topological space, S be a sequence of points of X. Recall that

A point x of X is said to be an accumulation point of S if every neighborhood of x contains a point of S different from x.

Show the following statements:

a) A point $x \in X$ is an accumulation point of S if and only if x is an accumulation point of the filter $\mathcal{F}_S$ associated with S, where $\mathcal{F}_S := \{A \subset X : |S \setminus A| < \infty\}$.

b) If S converges to some $x \in X$, then S is a relatively compact set in X.

c) If S converges to some $x \in X$, then $S \cup \{x\}$ is a bounded set in X.