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TOPOLOGICAL VECTOR SPACES II–WS 2019/2020

Exercise Sheet 1

This exercise sheet aims to assess your progress and to explicitly work out more details of some of the results proposed in the previous lectures. Please, hand in your solutions in postbox 18 near F411 by Wednesday the 13th of November at 13:30. The solutions to this assignment will be discussed in the tutorial on Tuesday the 19th of November (17:00–18:30) in P812.

1) Show the following statements which establish three methods to construct metrizable t.v.s..

a) If $\{X_j : j \in J\}$ is a countable family of metrizable t.v.s., then $\prod_{j\in J} X_j$ equipped with the product topology is also a metrizable t.v.s..

b) If X is a metrizable t.v.s. and $Y \subseteq X$ is a closed linear subspace, then the quotient space X/Y equipped with the quotient topology is metrizable t.v.s..

2) Prove the following general properties of metrizable t.v.s. (corresponding respectively to Propositions 1.1.6 and 1.1.7 in the lectures notes).

a) A metrizable t.v.s. X is complete if and only if X is sequentially complete.

b) Let X be a metrizable t.v.s. and Y be any t.v.s. (not necessarily metrizable). A mapping $f : X \to Y$ (not necessarily linear) is continuous if and only if it is sequentially continuous.

3) Consider the following space

$$\ell_1 := \left\{ x = (x_i)_{i\in\mathbb{N}} \subset \mathbb{R} : \sum_{i=1}^{\infty} |x_i| < \infty \right\}$$

endowed with the topology $\tau_{\mathcal{P}}$ induced by the family of seminorms $\mathcal{P} := \{p_n : n \in \mathbb{N}\}$, where for each $n \in \mathbb{N}$ the seminorm p_n is defined by

$$p_n(x) := \sum_{i=1}^{n} |x_i|, \qquad \forall x = (x_i)_{i\in\mathbb{N}} \in \ell_1.$$

Show that the t.v.s. $(\ell_1, \tau_{\mathcal{P}})$ is metrizable but *not* a Baire space. In particular, $(\ell_1, \tau_{\mathcal{P}})$ is not complete.

4) Let (X, τ_X) be a t.v.s., (Y, τ_Y) a Baire t.v.s. and $f : X \to Y$ a continuous linear map.

Show that if $f(X)$ has non-empty interior, then $\overline{f(V)} \in \mathcal{F}_{\tau_Y}(o)$ for each $V \in \mathcal{F}_{\tau_X}(o)$.