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TOPOLOGICAL VECTOR SPACES II–WS 2019/2020

Exercise Sheet 2

This exercise sheet aims to assess your progress and to explicitly work out more details of some of the results proposed in the previous lectures. Please, hand in your solutions in postbox 18 near F411 by Wednesday the 27th of November at 13:30. The solutions to this assignment will be discussed in the tutorial on Tuesday the 3rd of December (17:00–18:30) in P812.

1) Use Proposition 1 to give an example of a non-Hausdorff inductive topology on vector space E w.r.t. a family $\{(E_\alpha, \tau_\alpha, g_\alpha) : \alpha \in A\}$, where all the spaces (E_α, τ_α) are Hausdorff l.c. t.v.s. and all the maps $g_\alpha \colon E_\alpha \to E$ are linear.

Proposition 1. *Let τ_1 and τ_2 be two topologies on a vector space X such that (X, τ_i) is a Fréchet space for $i = 1, 2$. If $\tau_1 \cap \tau_2$ is a Hausdorff topology on X, then $\tau_1 = \tau_2$.*

2) Show that an $LF-$space E is a Baire space if and only if it is a Fréchet space.

3) Let E, F be two $LF-$spaces defined by the sequences $\{E_m\}_{m\in\mathbb{N}}$ and $\{F_n\}_{n\in\mathbb{N}}$, respectively. Prove the following statements:

a) If $u : E \to F$ is a continuous linear map, then for any $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $u(E_m) \subseteq F_n$.

b) If u is a topological isomorphism of E into F (i.e. bijective linear continuous and open), then for any $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $u^{-1}(F_n) \subseteq E_m$.

4) Let $(E, \|\cdot\|)$ be a normed space. For every $k \in \mathbb{N}$, let E_k be a linear subspace of E of dimension k, such that $E_k \subseteq E_{k+1}$. Let E_∞ be the union of all the E_k's equipped with the $LF-$topology defined by means of the sequence $\{E_k\}_{k\in\mathbb{N}}$.

Show that

$$V := \{x \in E_\infty : x \notin E_k \Rightarrow \|x\| < \tfrac{1}{k},\, k \in \mathbb{N}\}.$$

is not a neighbourhood of the origin in E_∞.

Hint: Show that V does not contain a convex neighbourhood of o.