Universität Konstanz
Fachbereich Mathematik und Statistik
Dr. Maria Infusino
Patrick Michalski

TOPOLOGICAL VECTOR SPACES II–WS 2019/2020

Exercise Sheet 3

This exercise sheet aims to assess your progress and to explicitly work out more details of some of the results proposed in the previous lectures. Please, hand in your solutions in postbox 18 near F411 by Wednesday the 11th of December at 13:30. The solutions to this assignment will be discussed in the tutorial on Tuesday the 17th of December (17:00–18:30) in P812.

1) Consider the space $\mathbb{R}[x_1, \dots, x_d]$ of polynomials in d real variables with real coefficients, provided with the LF-topology τ_{ind} introduced in Example I in Section 1.3 of the lecture notes. Prove the following two facts:

- **a)** The LF-topology τ_{ind} on $\mathbb{R}[x_1, \dots, x_d]$ is the finest locally convex topology on this space.
- **b)** Every linear map from $(\mathbb{R}[x_1, \dots, x_d], \tau_{\text{ind}})$ into any t.v.s. is continuous.

2) Consider the space $\mathcal{C}_c^\infty(\Omega)$ (with $\Omega \subseteq \mathbb{R}^d$ open) of test functions provided with the LF-topology τ_{ind} introduced in Example II in Section 1.3 of the lecture notes. Show that $(\mathcal{C}_c^\infty(\Omega), \tau_{\text{ind}})$ is not metrizable.

3) Let E be a vector space over $\mathbb{K}$ endowed with the projective topology τ_{proj} w.r.t. the family $\{(E_\alpha, \tau_\alpha, f_\alpha) : \alpha \in A\}$, where each (E_α, τ_α) is a locally convex t.v.s. over $\mathbb{K}$ and each f_α is a linear mapping from E to E_α. Let (F, τ) be an arbitrary t.v.s. and u be a linear mapping from F into E.

- **a)** Show that the mapping $u : F \to E$ is continuous if and only if, for each $\alpha \in A$, $f_\alpha \circ u : F \to E_\alpha$ is continuous.
- **b)** Does the previous statement still hold if we ignore the vector space structures, that is, if we just assume that E is a set, all (E_α, τ_α) and (F, τ) are topological spaces, each f_α is a mapping from E to E_α and τ_{proj} is the coarsest topology on E such that all mappings f_α are continuous?

4) Let $(A, \leq)$ be a directed partially ordered set and let (E, τ_{proj}) be the projective limit of the family $\{(E_\alpha, \tau_\alpha), \alpha \in A\}$ of l.c. t.v.s. w.r.t. the maps $\{g_{\alpha\beta} : \alpha, \beta \in A, \alpha \leq \beta\}$ and $\{f_\alpha : \alpha \in A\}$. Show that:

- **a)** $g_{\alpha\gamma} = g_{\alpha\beta} \circ g_{\beta\gamma}$ for all $\alpha \leq \beta \leq \gamma$ in A.
- **b)** $f_\alpha = g_{\alpha\beta} \circ f_\beta$ for all $\alpha \leq \beta$ in A.
- **c)** If (F, τ) is another locally convex t.v.s. and for each $\alpha \in A$, $g_\alpha \colon F \to E_\alpha$ is a continuous linear map such that $g_\alpha = g_{\alpha\beta} \circ g_\beta$ for all $\alpha \leq \beta$ in A, then there exists a unique continuous and linear map $g \colon F \to E$ such that $g_\alpha = f_\alpha \circ g$ for all $\alpha \in A$.