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TOPOLOGICAL VECTOR SPACES II–WS 2019/2020

Exercise Sheet 6

This exercise sheet aims to assess your progress and to explicitly work out more details of some of the results proposed in the previous lectures. Please, hand in your solutions in postbox 18 near F411 by Wednesday the 5th of February at 13:30. The solutions to this assignment will be discussed in the tutorial on Thursday the 13th of February (13:30–15:00) in D406.

1) Let $0 < p < 1$ and fix some $a, b \in \mathbb{R}$ with $a < b$. Consider the space $L^p([a,b])$ of all measurable functions $f : [a,b] \to \mathbb{R}$ such that $\int_a^b |f(t)|^p \,\mathrm{d}t < \infty$. Define a map

$$q_p(f) := \left(\int_a^b |f(t)|^p \,\mathrm{d}t \right)^{\frac{1}{p}} \quad \text{for all } f \in L^p([a,b])$$

and set $U(\varepsilon) := \{f \in L^p([a,b]) : q_p(f) \leq \varepsilon\}$ for $\varepsilon > 0$. Show that the following hold:

a) The sets $U(\varepsilon)$ with $\varepsilon > 0$ form a basis of neighbourhoods of the origin for a topology τ compatible with the vector space structure of $L^p([a,b])$.

Hint: Use the inequality $q_p(f+g) \leq 2^{\frac{1-p}{p}} (q_p(f) + q_p(g))$ for $f, g \in L^p([a,b])$.

b) The topological dual of $(L^p([a,b]), \tau)$ consists only of the zero functional.

2) Let Y be a closed linear subspace of a locally convex t.v.s. X.

Show that if $Y \neq X$, then there exists $f \in X'$ such that f is not identically zero on X but vanishes on Y.

3) Let Σ be a family of bounded subsets of a t.v.s. E such that the following two properties hold:

(P1) If $A, B \in \Sigma$, then there exists $C \in \Sigma$ such that $A \cup B \subseteq C$;

(P2) If $A \in \Sigma$ and $\lambda \in \mathbb{K}$, then there exits $B \in \Sigma$ such that $\lambda A \subseteq B$.

Show that:

a) The family $\mathcal{B} := \{W_\varepsilon(A) : A \in \Sigma, \varepsilon > 0\}$, where $W_\varepsilon(A) := \{x' \in E' : \sup_{x \in A} |\langle x', x \rangle| \leq \varepsilon\}$ for each $A \in \Sigma$ and each $\varepsilon > 0$, is a basis of neighbourhoods of the origin for the Σ-topology on E'.

b) The family of seminorms $\{p_A : A \in \Sigma\}$, where $p_A(x') := \sup_{x \in A} |\langle x', x \rangle|$ for each $A \in \Sigma$ and all $x' \in E'$, generates the Σ-topology on E'.

4) Given a t.v.s. E, show that a sequence $\{x'_n\}_{n \in \mathbb{N}}$ of elements in E' converges to the origin in the weak topology if and only if at each point $x \in E$ the sequence of their values $\{\langle x'_n, x \rangle\}_{n \in \mathbb{N}}$ converges to zero in $\mathbb{K}$.