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TOPOLOGICAL VECTOR SPACES II–WS 2019/2020

Recap Sheet 0

This recap sheet aims at refreshing your knowledge about the basic theory of topological vector spaces. You do **not** *need to hand in solutions, however if you should have any problem, please take advantage of the Fragestunde on Wednesday, November 6th, 10:00–11:30 in room F410 or on Thursday, November 7th, 11:45–13:15 in room F408.*

1) Let (X, τ) be a topological space.

a) Show that the family of neighbourhoods $\mathcal{F}_\tau(x)$ of a point $x \in X$ is a filter.

b) Let $S := (x_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence. Show that $\mathcal{F}_S := \{A \subseteq X : |S\backslash A| < \infty\}$ is a filter and $\mathcal{B} := \{S_m : m \in \mathbb{N}\}$ is a basis for $\mathcal{F}_S$, where $S_m := \{x_n \in S : n \geq m\}$ for all $m \in \mathbb{N}$.

2) Use Theorem 2.1.10 (TVS-I) to show that

a) Every normed space $(X, \|\cdot\|)$ endowed with the topology τ induced by $\|\cdot\|$, i.e. the family $\mathcal{B} := \{\{x \in X : \|x_0 - x\| < \varepsilon\} : x_0 \in X, \varepsilon > 0\}$ is a basis of τ, is a t.v.s..

b) $\mathbb{R}$ endowed with the lower limit topology τ, i.e. the family $\mathcal{B} := \{[a, b) : a < b \text{ in } \mathbb{R}\}$ is a basis of τ, is not a t.v.s..

3) Show that a t.v.s. (X, τ) is Hausdorff if and only if it is (T1), i.e. for each $x \in X\backslash\{o\}$ there exists $U \in \mathcal{F}_\tau(o)$ such that $x \notin U$. Conclude that (X, τ) is Hausdorff if and only if $\bigcap_{U\in\mathcal{F}_\tau(o)} U = \{o\}$ is closed.

4) Let (X, τ) be a t.v.s. and show that

a) For each $x \in X$ the filter of neighbourhoods $\mathcal{F}_\tau(x)$ is a Cauchy filter.

b) A filter finer than a Cauchy filter is a Cauchy filter.

c) Every converging filter is a Cauchy filter.

5) Let X be a vector space and $\mathcal{P}$ a family of seminorms on X. Show that the topology $\tau_\mathcal{P}$ induced by $\mathcal{P}$, i.e. $\tau_\mathcal{P}$ has

$$\mathcal{B} := \{\{x \in X : p_1(x) < \varepsilon, \ldots, p_n(x) < \varepsilon\} : n \in \mathbb{N}, p_1, \ldots, p_n \in \mathcal{P}, \varepsilon > 0\}$$

as a basis of neighbourhoods of the origin o in X, makes $(X, \tau_\mathcal{P})$ into a locally convex t.v.s. Recall that for any locally convex t.v.s. the topology is induced by a family of seminorms.