1.3 Inductive topologies and LF-spaces

Let $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ be a family of locally convex Hausdorff t.v.s. over the field $\mathbb{K}$ of real or complex numbers (A is an arbitrary index set). Let E be a vector space over the same field $\mathbb{K}$ and, for each $\alpha \in A$, let $g_\alpha : E_\alpha \to E$ be a linear mapping. The ***inductive topology*** τ_{ind} on E w.r.t. the family $\{(E_\alpha, \tau_\alpha, g_\alpha) : \alpha \in A\}$ is the topology generated by the following basis of neighbourhoods of the origin in E:

$$\begin{aligned}\mathcal{B}_{ind} : \ = \ & \{U \subset E \,\text{convex, balanced, absorbing} : \forall \alpha \in A,\ g_\alpha^{-1}(U) \text{ is} \\ & \text{a neighbourhood of the origin in}\,(E_\alpha, \tau_\alpha)\}.\end{aligned}$$

Note that τ_{ind} is the finest locally convex topology on E for which all the mappings g_α ($\alpha \in A$) are continuous. Suppose there exists a locally convex topology τ on E s.t. all the g_α's are continuous and $\tau_{ind} \subseteq \tau$. As (E, τ) is locally convex, there always exists a basis of neighbourhood of the origin consisting of convex, balanced, absorbing subsets of E. Then for any such a neighborhood U of the origin in (E, τ) we have, by continuity, that $g_\alpha^{-1}(U)$ is a neighbourhood of the origin in (E_α, τ_α). Hence, $U \in \mathcal{B}_{ind}$ and so $\tau \equiv \tau_{ind}$.

Proposition 1.3.1. *Let E be a vector space over $\mathbb{K}$ endowed with the inductive topology τ_{ind} w.r.t. the family $\{(E_\alpha, \tau_\alpha, g_\alpha) : \alpha \in A\}$, (F, τ) an arbitrary locally convex t.v.s., and u a linear mapping from E into F. The mapping $u : E \to F$ is continuous if and only if, for each $\alpha \in A$, $u \circ g_\alpha : E_\alpha \to F$ is continuous.*

Proof. Let W be a neighbourhood of the origin in (F, τ).

Suppose u is continuous, then we have that $u^{-1}(W)$ is a neighbourhood of the origin in (E, τ_{ind}). Therefore, there exists $U \in \mathcal{B}_{ind}$ s.t. $U \subseteq u^{-1}(W)$ and so

$$g_\alpha^{-1}(U) \subseteq g_\alpha^{-1}(u^{-1}(W)) = (u \circ g_\alpha)^{-1}(U), \quad \forall \alpha \in A. \tag{1.8}$$

As by definition of $\mathcal{B}_{ind}$, each $g_\alpha^{-1}(U)$ is a neighbourhood of the origin in (E_α, τ_α), so is $(u \circ g_\alpha)^{-1}(U)$ by (1.8). Hence, all $u \circ g_\alpha$ are continuous.

Conversely, suppose that for each $\alpha \in A$ the mapping $u \circ g_\alpha$ is continuous. As (F, τ) is locally convex, we can assume that W is a convex, balanced and absorbing neighbourhood of the origin in F. Then, by the linearity of u, we get that $u^{-1}(W)$ is a convex, balanced and absorbing subset of E. Moreover, the continuity of all $u \circ g_\alpha$ guarantees that each $(u \circ g_\alpha)^{-1}(W)$ is a neighbourhood of the origin in (E_α, τ_α), i.e. $g_\alpha^{-1}(u^{-1}(W))$ is a neighbourhood of the origin in (E_α, τ_α). Then $u^{-1}(W)$, being also convex, balanced and absorbing, must be in $\mathcal{B}_{ind}$ and so it is a neighborhood of the origin in (E, τ_{ind}). Hence, u is continuous. ☐

Let us consider now the case when we have a total order on the index set A and $\{E_\alpha : \alpha \in A\}$ is a family of linear subspaces of a vector space E over $\mathbb{K}$ which is directed under inclusions, i.e. $E_\alpha \subseteq E_\beta$ whenever $\alpha \leq \beta$, and s.t. $E = \cup_{\alpha\in A} E_\alpha$. For each $\alpha \in A$, let i_α be the canonical embedding of E_α in E and τ_α a topology on E_α s.t. (E_α, τ_α) is a locally convex Hausdorff t.v.s. and, whenever $\alpha \leq \beta$, the topology induced by τ_β on E_α is coarser than τ_α. The space E equipped with the inductive topology τ_{ind} w.r.t. the family $\{(E_\alpha, \tau_\alpha, i_\alpha) : \alpha \in A\}$ is said to be the ***inductive limit*** of the family of linear subspaces $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$.

An inductive limit of a family of linear subspaces $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ is said to be a ***strict inductive limit*** if, whenever $\alpha \leq \beta$, the topology induced by τ_β on E_α coincide with τ_α.

There are even more general constructions of inductive limits of a family of locally convex t.v.s. but in the following we will focus on a more concrete family of inductive limits which are more common in applications. Namely, we are going to consider the so-called ***LF-spaces***, i.e. countable strict inductive limits of increasing sequences of Fréchet spaces. For convenience, let us explicitly write down the definition of an LF-space.

Definition 1.3.2. *Let $\{E_n : n \in \mathbb{N}\}$ be an increasing sequence of linear subspaces of a vector space E over $\mathbb{K}$, i.e. $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$, such that $E = \cup_{n\in\mathbb{N}} E_n$. For each $n \in \mathbb{N}$ let (E_n, τ_n) be a Fréchet space such that the natural embedding i_n of E_n into E_{n+1} is a topological isomorphism, i.e. the topology induced by τ_{n+1} on E_n coincide with τ_n. The space E equipped with the inductive topology τ_{ind} w.r.t. the family $\{(E_n, \tau_n, i_n) : n \in \mathbb{N}\}$ is said to be the* LF-space with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$.

A basis of neighbourhoods of the origin in the LF-space (E, τ_{ind}) with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$ is given by:

$$\{U \subset E \text{ convex, balanced, abs.} : \forall n \in \mathbb{N},\, U \cap E_n \text{ is a nbhood of } o \text{ in } (E_n, \tau_n)\}.$$

In this case (E, τ_{ind}) is not only a locally convex t.v.s. but it is also Hausdorff.

Note that from the construction of the LF-space (E, τ_{ind}) with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$ we know that each E_n is isomorphically embedded in the subsequent ones, but a priori we do not know if E_n is isomorphically embedded in E, i.e. if the topology induced by τ_{ind} on E_n is identical to the topology τ_n initially given on E_n. This indeed true and it will be a consequence of the following lemma.

Lemma 1.3.3. *Let E be a locally convex space, E_0 a linear subspace of E equipped with the subspace topology, and U a convex neighborhood of the origin in E_0. Then there exists a convex neighborhood V of the origin in E such that $V \cap E_0 = U$.*

Proof.
As E_0 carries the subspace topology induced by E, there exists a neighborhood W of the origin in E such that $U = W \cap E_0$. Since E is a locally convex space, there exists a convex neighbourhood W_0 of the origin in E such that $W_0 \subseteq W$. Let V be the convex hull of $U \cup W_0$. Then by construction we have that V is a convex neighbourhood of the origin in E and that $U \subseteq V$ which implies $U = U \cap E_0 \subseteq V \cap E_0$. We claim that actually $V \cap E_0 = U$. Indeed, let $x \in V \cap E_0$; as $x \in V$ and as U and W_0 are both convex, we may write $x = ty + (1-t)z$ with $y \in U, z \in W_0$ and $t \in [0,1]$. If $t = 1$, then $x = y \in U$ and we are done. If $0 \leq t < 1$, then $z = (1-t)^{-1}(x - ty)$ belongs to E_0 and so $z \in W_0 \cap E_0 \subseteq W \cap E_0 = U$. This implies, by the convexity of U, that $x \in U$. Hence, $V \cap E_0 \subseteq U$. □

Proposition 1.3.4.
Let (E, τ_{ind}) be an LF-space with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$. Then

$$\tau_{ind} \restriction E_n \equiv \tau_n, \forall n \in \mathbb{N}.$$

Proof.
($\subseteq$) Let U be a neighbourhood of the origin in (E, τ_{ind}). Then, by definition of τ_{ind}, there exists V convex, balanced and absorbing neighbourhood of the origin in (E, τ_{ind}) s.t. $V \subseteq U$ and, for each $n \in \mathbb{N}$, $V \cap E_n$ is a neighbourhood of the origin in (E_n, τ_n). Hence, $\tau_{ind} \restriction E_n \subseteq \tau_n$, $\forall n \in \mathbb{N}$.

($\supseteq$) Given $n \in \mathbb{N}$, let U_n be a convex, balanced, absorbing neighbourhood of the origin in (E_n, τ_n). As E_n is a linear subspace of E, we can apply Lemma 1.3.3 which ensures the existence of a convex neighborhood U_{n+1} of the origin in E such that $U_{n+1} \cap E_n = U_n$. Then, by induction, we get that for any $k \in \mathbb{N}$ there exists a convex neighborhood U_{n+k} of the origin in E such that $U_{n+k} \cap E_{n+k-1} = U_{n+k-1}$. Hence, for any $k \in \mathbb{N}$ $U_{n+k} \cap E_n = U_n$. If we consider now $U := \cup_{k \in \mathbb{N}} U_{n+k}$, then $U \cap E_n = U_n$. Furthermore, U is a neighborhood of the origin in (E, τ_{ind}) since $U \cap E_m$ is a neighborhood of the origin in (E_m, τ_m) for all $m \in \mathbb{N}$. Hence, $\tau_n \subseteq \tau_{ind} \restriction E_n \, \forall n \in \mathbb{N}$. □

As a particular case of Proposition 1.3.1 we get that:

Proposition 1.3.5.
Let (E, τ_{ind}) be an LF-space with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$ and (F, τ) an arbitrary locally convex t.v.s..

1. *A linear mapping u from E into F is continuous if and only if, for each $n \in \mathbb{N}$, the restriction $u \restriction E_n$ of u to E_n is continuous.*
2. *A linear form on E is continuous if and only if its restrictions to each E_n are continuous.*

Note that Propositions 1.3.4 and 1.3.5 hold for any countable strict inductive limit of an increasing sequences of locally convex Hausdorff t.v.s. (even when they are not Fréchet). The following results is instead typical of LF-spaces as it heavily relies on the completeness of the t.v.s. of the defining sequence.

Theorem 1.3.6. *Any LF-space is complete.*

Proof.
Let (E, τ_{ind}) be an LF-space with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$. Let $\mathcal{F}$ be a Cauchy filter in E. Denote by $\mathcal{F}_E(o)$ the filter of neighbourhoods of the origin in E and consider

$$\mathcal{G} := \{A \subseteq E : A \supseteq M + V \text{ for some } M \in \mathcal{F}, V \in \mathcal{F}_E(o)\}.$$

1) $\mathcal{G}$ is a filter on E.
Indeed, it is clear from its definition that $\mathcal{G}$ does not contain the empty set and that any subset of E containing a set in $\mathcal{G}$ has to belong to $\mathcal{G}$. Moreover, for any $A_1, A_2 \in \mathcal{G}$ there exist $M_1, M_2 \in \mathcal{F}$, $V_1, V_2 \in \mathcal{F}_E(o)$ s.t. $M_1 + V_1 \subseteq A_1$ and $M_2 + V_2 \subseteq A_2$; and therefore

$$A_1 \cap A_2 \supseteq (M_1 + V_1) \cap (M_2 + V_2) \supseteq (M_1 \cap M_2) + (V_1 \cap V_2).$$

The latter proves that $A_1 \cap A_2 \in \mathcal{G}$ since $\mathcal{F}$ and $\mathcal{F}_E(o)$ are both filters and so $M_1 \cap M_2 \in \mathcal{F}$ and $V_1 \cap V_2 \in \mathcal{F}_E(o)$.

2) $\mathcal{G} \subseteq \mathcal{F}$.
In fact, for any $A \in \mathcal{G}$ there exist $M \in \mathcal{F}$ and $V \in \mathcal{F}_E(o)$ s.t.

$$A \supseteq M + V \supset M + \{0\} = M$$

which implies that $A \in \mathcal{F}$ since $\mathcal{F}$ is a filter.

3) $\mathcal{G}$ is a Cauchy filter on E.
Let $U \in \mathcal{F}_E(o)$. Then there always exists $V \in \mathcal{F}_E(o)$ balanced such that

$V + V - V \subseteq U$. As $\mathcal{F}$ is a Cauchy filter on E, there exists $M \in \mathcal{F}$ such that $M - M \subseteq V$. Then

$$(M + V) - (M + V) \subseteq (M - M) + (V - V) \subseteq V + V - V \subseteq U$$

which proves that $\mathcal{G}$ is a Cauchy filter since $M + V \in \mathcal{G}$.

It is possible to show (and we do it later on) that:

$$\exists\, p \in \mathbb{N} : \forall A \in \mathcal{G},\; A \cap E_p \neq \emptyset \tag{1.9}$$

This property ensures that the family

$$\mathcal{G}_p := \{A \cap E_p : \; A \in \mathcal{G}\}$$

is a filter on E_p. Moreover, since $\mathcal{G}$ is a Cauchy filter on E and since by Proposition 1.3.4 we have $\tau_{ind} \restriction E_p = \tau_p$, $\mathcal{G}_p$ is a Cauchy filter on E_p. Hence, the completeness of E_p guarantees that there exists $x \in E_p$ s.t. $\mathcal{G}_p \to x$. This implies that also $\mathcal{G} \to x$ and so $\mathcal{F}_E(o) \subseteq \mathcal{G} \subseteq \mathcal{F}$ which gives $\mathcal{F} \to x$. □

Proof. of (1.9)
Suppose that (1.9) is false, i.e. $\forall n \in \mathbb{N}$, $\exists\, A_n \in \mathcal{G}$ s.t. $A_n \cap E_n = \emptyset$. By the definition of $\mathcal{G}$, this means that

$$\forall n \in \mathbb{N},\; \exists\, M_n \in \mathcal{F},\; V_n \in \mathcal{F}_E(o), \text{ s.t. } (M_n + V_n) \cap E_n = \emptyset. \tag{1.10}$$

Since E is a locally convex t.v.s., we may assume that each V_n is balanced and convex, and that $V_{n+1} \subseteq V_n$ for all $n \in \mathbb{N}$. Consider

$$W_n := conv\left(V_n \cup \bigcup_{k=1}^{n-1} (V_k \cap E_k)\right),$$

then

$$(W_n + M_n) \cap E_n = \emptyset, \forall n \in \mathbb{N}.$$

Indeed, if there exists $h \in (W_n + M_n) \cap E_n$ then $h \in E_n$ and $h \in (W_n + M_n)$. We may then write: $h = x + ty + (1-t)z$ with $x \in M_n$, $y \in V_n$, $z \in V_1 \cap E_{n-1}$ and $t \in [0,1]$. Hence, $x + ty = h - (1-t)z \in E_n$. But we also have $x + ty \in M_n + V_n$, since V_n is balanced and so $ty \in V_n$. Therefore, $x + ty \in (M_n + V_n) \cap E_n$ which contradicts (1.10).

Now let us define

$$W := conv\left(\bigcup_{k=1}^{\infty} (V_k \cap E_k)\right).$$

As W is convex and as $W \cap E_k$ contains $V_k \cap E_k$ for all $k \in \mathbb{N}$, W is a neighborhood of the origin in (E, τ_{ind}). Moreover, as $(V_n)_{n\in\mathbb{N}}$ is decreasing, we have that for all $n \in \mathbb{N}$

$$W \subseteq \bigcup_{k=1}^{n} (V_k \cap E_k) \subseteq V_n \cup \bigcup_{k=1}^{n-1} (V_k \cap E_k) = W_n.$$

Since $\mathcal{F}$ is a Cauchy filter, there exists $B \in \mathcal{F}$ such that $B - B \subseteq W$ and so $B - B \subseteq W_n, \forall n \in \mathbb{N}$. On the other hand we have $B \cap M_n \neq \emptyset, \forall n \in \mathbb{N}$, as both B and M_n belong to $\mathcal{F}$. Hence, for all $n \in \mathbb{N}$ we get

$$B - (B \cap M_n) \subseteq B - B \subseteq W_n,$$

which implies

$$B \subseteq W_n + (B \cap M_n) \subseteq W_n + M_n$$

and so

$$B \cap E_n \subseteq (W_n + M_n) \cap E_n = \emptyset.$$

Therefore, we have got that $B \cap E_n = \emptyset$ for all $n \in \mathbb{N}$ and so that $B = \emptyset$, which is impossible as $B \in \mathcal{F}$. □