1.3. Inductive topologies and LF-spaces

As $W$ is convex and as $W \cap E_k$ contains $V_k \cap E_k$ for all $k \in \mathbb{N}$, $W$ is a neighborhood of the origin in $(E, \tau_{ind})$. Moreover, as $(V_n)_{n \in \mathbb{N}}$ is decreasing, we have that for all $n \in \mathbb{N}$

$$W \subseteq \bigcup_{k=1}^{n}(V_k \cap E_k) \subseteq V_n \cup \bigcup_{k=1}^{n-1}(V_k \cap E_k) = W_n.$$ 

Since $\mathcal{F}$ is a Cauchy filter, there exists $B \in \mathcal{F}$ such that $B - B \subseteq W$ and so $B - B \subseteq W_n$, $\forall n \in \mathbb{N}$. On the other hand we have $B \cap M_n \neq \emptyset$, $\forall n \in \mathbb{N}$, as both $B$ and $M_n$ belong to $\mathcal{F}$. Hence, for all $n \in \mathbb{N}$ we get

$$B - (B \cap M_n) \subseteq B - B \subseteq W_n,$$

which implies

$$B \subseteq W_n + (B \cap M_n) \subseteq W_n + M_n$$

and so

$$B \cap E_n \subseteq (W_n + M_n) \cap E_n = \emptyset.$$ 

Therefore, we have got that $B \cap E_n = \emptyset$ for all $n \in \mathbb{N}$ and so that $B = \emptyset$, which is impossible as $B \in \mathcal{F}$.

Example I: The space of polynomials

Let $n \in \mathbb{N}$ and $\mathbf{x} := (x_1, \ldots, x_n)$. Denote by $\mathbb{R}[\mathbf{x}]$ the space of polynomials in the $n$ variables $x_1, \ldots, x_n$ with real coefficients. A canonical algebraic basis for $\mathbb{R}[\mathbf{x}]$ is given by all the monomials

$$\mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \forall \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n.$$ 

For any $d \in \mathbb{N}_0$, let $\mathbb{R}_d[\mathbf{x}]$ be the linear subspace of $\mathbb{R}[\mathbf{x}]$ spanned by all monomials $\mathbf{x}^\alpha$ with $|\alpha| := \sum_{i=1}^{n} \alpha_i \leq d$, i.e.

$$\mathbb{R}_d[\mathbf{x}] := \{ f \in \mathbb{R}[\mathbf{x}] | \deg f \leq d \}.$$ 

Since there are exactly $\binom{n+d}{d}$ monomials $\mathbf{x}^\alpha$ with $|\alpha| \leq d$, we have that

$$\dim(\mathbb{R}_d[\mathbf{x}]) = \frac{(d+n)!}{d!n!},$$

and so that $\mathbb{R}_d[\mathbf{x}]$ is a finite dimensional vector space. Hence, by Tychonoff Theorem (see Corollary 3.1.4 in TVS-I) there is a unique topology that makes $\mathbb{R}_d[\mathbf{x}]$ into a Hausdorff t.v.s. which is also complete and so Fréchet (as it topologically isomorphic to $\mathbb{R}^{\dim(\mathbb{R}_d[\mathbf{x}])}$ equipped with the euclidean topology $\tau^d_e$).
1. **Special classes of topological vector spaces**

As \( \mathbb{R}[x] := \bigcup_{d=0}^{\infty} \mathbb{R}_d[x] \), we can then endow it with the inductive topology \( \tau_{ind} \) w.r.t. the family of F-spaces \( \{ \mathbb{R}_d[x], d \in \mathbb{N}_0 \} \); thus \((\mathbb{R}[x], \tau_{ind})\) is a LF-space and the following properties hold (proof as Exercise 1, Sheet 3):

- a) \( \tau_{ind} \) is the finest locally convex topology on \( \mathbb{R}[x] \),
- b) every linear map \( f \) from \((\mathbb{R}[x], \tau_{ind})\) into any t.v.s. is continuous.

**Example II: The space of test functions**

Let \( \Omega \subseteq \mathbb{R}^d \) be open in the euclidean topology. For any integer \( 0 \leq k \leq \infty \), we have defined in Section 1.2 the set \( C^k(\Omega) \) of all real valued \( k \)-times continuously differentiable functions on \( \Omega \), which is a real vector space w.r.t. pointwise addition and scalar multiplication. We have equipped this space with the \( C^k \)-topology (i.e. the topology of uniform convergence on compact sets of the functions and their derivatives up to order \( k \)) and showed that this turns \( C^k(\Omega) \) into a Fréchet space.

Let \( K \) be a compact subset of \( \Omega \), which means that it is bounded and closed in \( \mathbb{R}^d \) and that its closure is contained in \( \Omega \). For any integer \( 0 \leq k \leq \infty \), consider the subset \( C^k_c(K) \) of \( C^k(\Omega) \) consisting of all the functions \( f \in C^k(\Omega) \) whose support lies in \( K \), i.e.

\[
C^k_c(K) := \{ f \in C^k(\Omega) : \text{supp}(f) \subseteq K \},
\]

where \( \text{supp}(f) \) denotes the support of the function \( f \) on \( \Omega \), that is the closure in \( \Omega \) of the subset \( \{ x \in \Omega : f(x) \neq 0 \} \).

For any integer \( 0 \leq k \leq \infty \), \( C^k_c(K) \) is always a closed linear subspace of \( C^k(\Omega) \). Indeed, for any \( f, g \in C^k_c(K) \) and any \( \lambda \in \mathbb{R} \), we clearly have \( f + g \in C^k_c(K) \) and \( \lambda f \in C^k_c(K) \) and also \( \text{supp}(f + g) \subseteq \text{supp}(f) \cup \text{supp}(g) \subseteq K \) and \( \text{supp}(\lambda f) = \text{supp}(f) \subseteq K \), which gives \( f + g, \lambda f \in C^k_c(K) \). To show that \( C^k_c(K) \) is closed in \( C^k(\Omega) \), it suffices to prove that it is sequentially closed as \( C^k(\Omega) \) is a F-space. Consider a sequence \( (f_j)_{j \in \mathbb{N}} \) of functions in \( C^k_c(K) \) converging to \( f \) in the \( C^k \)-topology. Then clearly \( f \in C^k(\Omega) \) and since all the \( f_j \) vanish in the open set \( \Omega \setminus K \), obviously their limit \( f \) must also vanish in \( \Omega \setminus K \). Thus, regarded as a subspace of \( C^k(\Omega) \), \( C^k_c(K) \) is also complete (see Proposition 2.5.8 in TVS-I) and so it is itself an F-space.

Let us now denote by \( C^k_c(\Omega) \) the union of the subspaces \( C^k_c(K) \) as \( K \) varies in all possible ways over the family of compact subsets of \( \Omega \), i.e. \( C^k_c(\Omega) \) is linear subspace of \( C^k(\Omega) \) consisting of all the functions belonging to \( C^k(\Omega) \) which have a compact support (this is what is actually encoded in the subscript \( c \)). In particular, the space \( C^\infty_c(\Omega) \) (smooth functions with compact support in \( \Omega \)) is called space of test functions and plays an essential role in the theory of distributions.
We will not endow \( C^k_c(\Omega) \) with the subspace topology induced by \( C^k(\Omega) \), but we will consider a finer one, which will turn \( C^k_c(\Omega) \) into an LF-space. Let us consider a sequence \((K_j)_{j \in \mathbb{N}}\) of compact subsets of \( \Omega \) s.t. \( K_j \subseteq K_{j+1}, \forall j \in \mathbb{N} \) and \( \bigcup_{j=1}^{\infty} K_j = \Omega \). (Sometimes is even more advantageous to choose the \( K_j \)'s to be relatively compact i.e. the closures of open subsets of \( \Omega \) such that \( K_j \subseteq K_{j+1}, \forall j \in \mathbb{N} \) and \( \bigcup_{j=1}^{\infty} K_j = \Omega \).)

Then \( C^k_c(\Omega) = \bigcup_{j=1}^{\infty} C^k_c(K_j) \), as an arbitrary compact subset \( K \) of \( \Omega \) is contained in \( K_j \) for some sufficiently large \( j \). Because of our way of defining the F-spaces \( C^k_c(K_j) \), we have that \( C^k_c(K_{j+1}) \) induces on the subset \( C^k_c(K_j) \) the same topology as the one originally given on it, i.e. the subspace topology induced on \( C^k_c(K_j) \) by \( C^k(\Omega) \). Thus we can equip \( C^k_c(\Omega) \) with the inductive topology \( \tau_{\text{ind}} \) w.r.t. the sequence of F-spaces \( \{C^k_c(K_j), j \in \mathbb{N}\} \), which makes \( C^k_c(\Omega) \) an LF-space. It is easy to check that this topology does not depend on the choice of the sequence of compact sets \( K_j \)'s provided they fill \( \Omega \).

**Proposition 1.3.7.** For any integer \( 0 \leq k \leq \infty \), consider \( C^k_c(\Omega) \) endowed with the LF-topology \( \tau_{\text{ind}} \) described above. Then we have the following continuous injections:

\[
C^\infty_c(\Omega) \to C^k_c(\Omega) \to C^{k-1}_c(\Omega), \quad \forall 0 < k < \infty.
\]

**Proof.** Let us just prove the first inclusion \( i : C^\infty_c(\Omega) \to C^k_c(\Omega) \) as the others follows in the same way. As \( C^\infty_c(\Omega) = \bigcup_{j=1}^{\infty} C^\infty_c(K_j) \) is the inductive limit of the sequence of F-spaces \( \{C^\infty_c(K_j)\}_{j \in \mathbb{N}} \), where \( (K_j)_{j \in \mathbb{N}} \) is a sequence of compact subsets of \( \Omega \) such that \( K_j \subseteq K_{j+1}, \forall j \in \mathbb{N} \) and \( \bigcup_{j=1}^{\infty} K_j = \Omega \), by Proposition 1.3.5 we know that \( i \) is continuous if and only if, for any \( j \in \mathbb{N} \), \( e_j := i \upharpoonright C^\infty_c(K_j) \) is continuous. But from the definition we gave of the topology on each \( C^k_c(K_j) \) and \( C^\infty_c(K_j) \), it is clear that both the inclusions \( i_j : C^\infty_c(K_j) \to C^k_c(K_j) \) and \( s_j : C^k_c(K_j) \to C^k_c(\Omega) \) are continuous. Hence, for each \( j \in \mathbb{N} \), \( e_j = s_j \circ i_j \) is indeed continuous. \( \square \)

### 1.4 Projective topologies and examples of projective limits

Let \( \{(E_\alpha, \tau_\alpha) : \alpha \in A\} \) be a family of locally convex t.v.s. over the field \( \mathbb{K} \) of real or complex numbers (\( A \) is an arbitrary index set). Let \( E \) be a vector space over the same field \( \mathbb{K} \) and, for each \( \alpha \in A \), let \( f_\alpha : E \to E_\alpha \) be a linear mapping. The **projective topology** \( \tau_{\text{proj}} \) on \( E \) w.r.t. the family \( \{(E_\alpha, \tau_\alpha, f_\alpha) : \alpha \in A\} \) is the coarsest topology on \( E \) for which all the mappings \( g_\alpha (\alpha \in A) \) are continuous.
1. Special classes of topological vector spaces

A basis of neighbourhoods of a point \( x \in E \) is given by:

\[
B_{\text{proj}}(x) := \left\{ \bigcap_{\alpha \in F} f_\alpha^{-1}(U_\alpha) : F \subseteq A \text{ finite}, U_\alpha \text{ neighborhood of } f_\alpha(x) \text{ in } (E_\alpha, \tau_\alpha), \forall \alpha \in F \right\}.
\]

Since the \( f_\alpha \) are linear mappings and the \((E_\alpha, \tau_\alpha)\) are locally convex t.v.s., \( \tau_{\text{proj}} \) is a translation-invariant topology on \( E \) with a base of convex, balanced and absorbing neighborhoods of the origin satisfying conditions of Theorem 4.1.14 in TVS-I; hence \( \tau_{\text{proj}} \) is a locally convex topology on \( E \).

**Proposition 1.4.1.** The projective topology \( \tau_{\text{proj}} \) on \( E \) w.r.t. the family \( \{(E_\alpha, \tau_\alpha, f_\alpha) : \alpha \in A\} \) is a Hausdorff topology if and only if for each \( 0 \neq x \in E \), there exists an \( \alpha \in A \) and a neighbourhood of the origin in \((E_\alpha, \tau_\alpha)\) such that \( f_\alpha(x) \notin U_\alpha \).

**Proof.** Suppose that \((E, \tau_{\text{proj}})\) is Hausdorff and let \( 0 \neq x \in E \). By Proposition 2.2.3 in TVS-I, there exists a neighborhood \( U \) of the origin in \( E \) not containing \( x \). Then, by definition of \( \tau_{\text{proj}} \) there exists a finite subset \( F \subseteq A \) and, for any \( \alpha \in F \), there exists \( U_\alpha \) neighbourhood of the origin in \((E_\alpha, \tau_\alpha)\) s.t. \( \bigcap_{\alpha \in F} f_\alpha^{-1}(U_\alpha) \subseteq U \). Hence, as \( x \notin U \), there exists \( \alpha \in F \) s.t. \( x \notin f_\alpha^{-1}(U_\alpha) \) i.e. \( f_\alpha(x) \notin U_\alpha \).

Conversely, suppose that there exists an \( \alpha \in A \) and a neighbourhood of the origin in \((E_\alpha, \tau_\alpha)\) such that \( f_\alpha(x) \notin U_\alpha \). Then \( x \notin f_\alpha^{-1}(U_\alpha) \), which implies by Proposition 2.2.3 in TVS-I that \( \tau_{\text{proj}} \) is a Hausdorff topology, as \( f_\alpha^{-1}(U_\alpha) \) is a neighbourhood of the origin in \((E, \tau_{\text{proj}})\) not containing \( x \).

**Proposition 1.4.2.** Let \( E \) be a vector space over \( \mathbb{K} \) endowed with the projective topology \( \tau_{\text{proj}} \) w.r.t. the family \( \{(E_\alpha, \tau_\alpha, f_\alpha) : \alpha \in A\} \), where each \((E_\alpha, \tau_\alpha)\) is a locally convex t.v.s. over \( \mathbb{K} \) and each \( f_\alpha \) a linear mapping from \( E \) to \( E_\alpha \). Let \((F, \tau)\) be an arbitrary t.v.s. and \( u \) a linear mapping from \( F \) into \( E \). The mapping \( u : F \to E \) is continuous if and only if, for each \( \alpha \in A \), \( f_\alpha \circ u : F \to E_\alpha \) is continuous.

**Proof.** (Sheet 3, Exercise 2)

**Example I: The product of locally convex t.v.s**

Let \( \{(E_\alpha, \tau_\alpha) : \alpha \in A\} \) be a family of locally convex t.v.s. The product topology \( \tau_{\text{proj}} \) on \( E = \prod_{\alpha \in A} E_\alpha \) (see Definition 1.1.18 in TVS-I) is evidently locally convex and it is the coarsest topology for which all the canonical projections \( p_\alpha : E \to E_\alpha \) are continuous. Hence, the product topology coincide with the projective topology on \( E \) with respect to \( \{(E_\alpha, \tau_\alpha, p_\alpha) : \alpha \in A\} \).
Let us consider now the case when we have a total order on the index set $A$, $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ is a family of locally convex t.v.s. over $\mathbb{K}$ and for any $\alpha \leq \beta$ we have a continuous linear mapping $g_{\alpha \beta} : E_\beta \to E_\alpha$. Let $E$ be the subspace of $\prod_{\alpha \in A} E_\alpha$ whose elements $x = (x_\alpha)_{\alpha \in A}$ satisfy the relation $x_\alpha = g_{\alpha \beta}(x_\beta)$ whenever $\alpha \leq \beta$. For any $\alpha \in A$, let $f_\alpha$ be the canonical projection $p_\alpha : \prod_{\alpha \in A} E_\alpha \to E_\alpha$ restricted to $E$. The space $E$ endowed with the projective topology w.r.t. the family $\{(E_\alpha, \tau_\alpha, f_\alpha) : \alpha \in A\}$ is said to be the projective limit of the family $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ w.r.t. the mappings $\{g_{\alpha \beta} : \alpha, \beta \in A, \alpha \leq \beta\}$. If each $f_\alpha(E)$ is dense in $E_\alpha$ then the projective limit is said to be reduced.

Remark 1.4.3. There are even more general constructions of projective limits of a family of locally convex t.v.s. (even when the index set is not ordered) but in the following we will focus on a particular kind of reduced projective limits. Namely, given an index set $A$, and a family $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ of locally convex t.v.s. over $\mathbb{K}$ which is directed by topological embeddings (i.e. for any $\alpha, \beta \in A$ there exists $\gamma \in A$ s.t. $E_\gamma \subset E_\alpha$ and $E_\gamma \subset E_\beta$) and such that the set $E := \bigcap_{\alpha \in A} E_\alpha$ is dense in each $E_\alpha$, we will consider the reduced projective limit $(E, \tau_{\text{proj}})$. Here, $\tau_{\text{proj}}$ is the projective topology w.r.t. the family $\{(E_\alpha, \tau_\alpha, i_\alpha) : \alpha \in A\}$, where each $i_\alpha$ is the embedding of $E$ into $E_\alpha$.

Example II: The space of test functions
Let $\Omega \subseteq \mathbb{R}^d$ be open in the euclidean topology. The space of test functions $C^\infty_c(\Omega)$, i.e. the space of all the functions belonging to $C^\infty(\Omega)$ which have a compact support, can be constructed as reduced projective limit of the kind introduced in Remark 1.4.3.

Consider the index set

$$T := \{t := (t_1, t_2) : t_1 \in \mathbb{N}_0, t_2 \in C^\infty(\Omega) \text{ with } t_2(x) \geq 1, \forall x \in \Omega\}$$

and for each $t \in T$, let us introduce the following norm on $C^\infty_c(\Omega)$:

$$\|\varphi\|_t := \sup_{x \in \Omega} \left( t_2(x) \sum_{|\alpha| \leq t_1} |(D^\alpha \varphi)(x)| \right).$$

For each $t \in T$, let $\mathcal{D}_t(\Omega)$ be the completion of $C^\infty_c(\Omega)$ w.r.t. $\|\cdot\|_t$. Then as sets

$$C^\infty_c(\Omega) = \bigcap_{t \in T} \mathcal{D}_t(\Omega).$$
1. Special classes of topological vector spaces

The space of test functions $\mathcal{C}_c^\infty(\Omega)$ endowed with the projective topology w.r.t. the family $\{(\mathcal{D}(\Omega), \tau_t, i_t) : t \in T\}$, where (for each $t \in T$) $\tau_t$ denotes the topology induced by the norm $\| \cdot \|_t$ and $i_t$ denotes the natural embedding of $\mathcal{C}_c^\infty(\Omega)$ into $\mathcal{D}(\Omega)$.

Using Sobolev embeddings theorems, it can be showed that the space of test functions $\mathcal{C}_c^\infty(\Omega)$ can be actually written as projective limit of a family of weighted Sobolev spaces which are Hilbert spaces (see Chapter I, Section 3.10 of the book [Y. M. Berezansky, Selfadjoint Operators in Spaces of Functions of Infinite Many Variables, vol. 63, Trans. Amer. Math. Soc., 1986]).