Introduction

The theory of topological vector spaces (TVS), as the name suggests, is a beautiful connection between topological and algebraic structures. The first systematic treatment of these spaces appeared in “Livre V: Espaces vectoriels topologiques (1953)” in the series “Éléments de mathématique” by Nicolas Bourbaki. Actually, there was no person called Nicolas Bourbaki but this was just a pseudonym under which a group of mathematicians wrote the above mentioned series of books between 1935 and 1983 with the aim of reformulating the whole mathematics on an extremely formal, rigourous and general basis grounded on set theory. The work of the Bourbaki group (officially known as the “Association of collaborators of Nicolas Bourbaki”) greatly influenced the mathematic world and led to the discovery of concepts and terminologies still used today (e.g. the symbol $\emptyset$, the notions of injective, surjective, bijective, etc.) The Bourbaki group included several mathematicians connected to the École Normale Supérieure in Paris such as Henri Cartan, Jean Coulomb, Jean Dieudonné, André Weil, Laurent Schwartz, Jean-Pierre Serre, Alexander Grothendieck. The latter is surely the name which is most associated to the theory of TVS. Of course great contributions to this theory were already given before him (e.g. the Banach and Hilbert spaces are examples of TVS), but Alexander Grothendieck was engaged in a completely general approach to the study of these spaces and collected some among the deepest results on TVS in his Phd thesis (1950-1953) written under the supervision of Jean Dieudonné and Laurent Schwartz. After his dissertation he said: “There is nothing more to do, the subject is dead”. Despite this sentence come out of the mouth of a genius, the theory of TVS is far from being dead. Many aspects are in fact still unknown and the theory lively interacts with several interesting problems which are still currently unsolved!

Chapter 1

Preliminaries

1.1 Topological spaces

1.1.1 The notion of topological space

The topology on a set X is usually defined by specifying its open subsets of X. However, in dealing with topological vector spaces, it is often more convenient to define a topology by specifying what the neighbourhoods of each point are.

Definition 1.1.1. *A* topology τ *on a set* X *is a family of subsets of* X *which satisfies the following conditions:*
(O1) *the empty set* $\emptyset$ *and the whole* X *are both in* τ
(O2) τ *is closed under finite intersections*
(O3) τ *is closed under arbitrary unions*
The pair (X, τ) *is called a* topological space.

The sets $O \in \tau$ are called *open sets* of X and their complements $C = X \setminus O$ are called *closed sets* of X. A subset of X may be neither closed nor open, either closed or open, or both. A set that is both closed and open is called a *clopen set*.

Definition 1.1.2. *Let* (X, τ) *be a topological space.*

- *A subfamily* $\mathcal{B}$ *of* τ *is called a* basis *if every open set can be written as a union (possibly empty) of sets in* $\mathcal{B}$.
- *A subfamily* $\mathcal{X}$ *of* τ *is called a* subbasis *if the finite intersections of its sets form a basis, i.e. every open set can be written as a union of finite intersections of sets in* $\mathcal{X}$.

Therefore, a topology τ on X is completely determined by a basis or a subbasis.

Examples 1.1.3.

a) *The family* $\mathcal{B} := \{(a,b) : a, b \in \mathbb{Q}$ *with* $a < b\}$ *is a basis of the euclidean (or standard) topology on* $\mathbb{R}$.

b) *The collection* $\mathcal{S}$ *of all semi-infinite intervals of the real line of the forms* $(-\infty, a)$ *and* $(a, +\infty)$, *where* $a \in \mathbb{R}$ *is not a base for any topology on* $\mathbb{R}$. *To show this, suppose it were. Then, for example,* $(-\infty, 1)$ *and* $(0, \infty)$ *would be in the topology generated by* S, *being unions of a single base element, and so their intersection* $(0, 1)$ *would be by the axiom (O2) of topology. But* $(0, 1)$ *clearly cannot be written as a union of elements in* $\mathcal{S}$.
However, $\mathcal{S}$ *is a subbasis of the euclidean topology on* $\mathbb{R}$.

Proposition 1.1.4. *Let* X *be a set and let* $\mathcal{B}$ *be a collection of subsets of* X. $\mathcal{B}$ *is a basis for a topology* τ *on* X *iff the following hold:*

1. $\mathcal{B}$ *covers* X, *i.e.* $\forall\, x \in X$, $\exists\, B \in \mathcal{B}$ *s.t.* $x \in B$.
2. *If* $x \in B_1 \cap B_2$ *for some* $B_1, B_2 \in \mathcal{B}$, *then* $\exists\, B_3 \in \mathcal{B}$ *s.t.* $x \in B_3 \subseteq B_1 \cap B_2$.

Proof. (Sheet 1, Exercise 1 a)) □

Definition 1.1.5. *Let* (X, τ) *be a topological space and* $x \in X$. *A subset* U *of* X *is called a* neighbourhood *of* x *if it contains an open set containing the point* x, *i.e.* $\exists\, O \in \tau$ *s.t.* $x \in O \subseteq U$. *The family of all neighbourhoods of a point* $x \in X$ *is denoted by* $\mathcal{F}(x)$.

In order to define a topology on a set by the family of neighbourhoods of each of its points, it is convenient to introduce the notion of filter. Note that the notion of filter is given on a set which does not need to carry any other structure. Thus this notion is perfectly independent of the topology.

Definition 1.1.6. *A* filter *on a set* X *is a family* $\mathcal{F}$ *of subsets of* X *which fulfills the following conditions:*

(F1) *the empty set* $\emptyset$ *does not belong to* $\mathcal{F}$
(F2) $\mathcal{F}$ *is closed under finite intersections*
(F3) *any subset of* X *containing a set in* $\mathcal{F}$ *belongs to* $\mathcal{F}$

Definition 1.1.7. *A family* $\mathcal{B}$ *of subsets of* X *is called a* basis of a filter $\mathcal{F}$ *if*

1. $\mathcal{B} \subseteq \mathcal{F}$
2. $\forall A \in \mathcal{F}$, $\exists\, B \in \mathcal{B}$ *s.t.* $B \subseteq A$

Examples 1.1.8.

a) *The family* $\mathcal{G}$ *of all subsets of a set* X *containing a fixed non-empty subset* A *is a filter and* $\mathcal{B} = \{A\}$ *is its base.* $\mathcal{G}$ *is called the* principle filter *generated by* A.

b) *Given a topological space* X *and* $x \in X$, *the family* $\mathcal{F}(x)$ *is a filter.*

c) *Let $S := \{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in a set X. Then the family $\mathcal{F} := \{A \subset X : |S \setminus A| < \infty\}$ is a filter and it is known as the* filter associated *to S. For each $m \in \mathbb{N}$, set $S_m := \{x_n \in S : n \geq m\}$. Then $\mathcal{B} := \{S_m : m \in \mathbb{N}\}$ is a basis for $\mathcal{F}$.*

Proof. (Sheet 1, Exercise 2). □

Theorem 1.1.9. *Given a topological space X and a point $x \in X$, the filter of neighbourhoods $\mathcal{F}(x)$ satisfies the following properties.*
(N1) *For any $A \in \mathcal{F}(x)$, $x \in A$.*
(N2) *For any $A \in \mathcal{F}(x)$, $\exists\, B \in \mathcal{F}(x)$: $\forall\, y \in B$, $A \in \mathcal{F}(y)$.*
Viceversa, if for each point x in a set X we are given a filter $\mathcal{F}_x$ fulfilling the properties (N1) and (N2) then there exists a unique topology τ s.t. for each $x \in X$, $\mathcal{F}_x$ is the family of neighbourhoods of x, i.e. $\mathcal{F}_x \equiv \mathcal{F}(x)$, $\forall x \in X$.

This means that a topology on a set is uniquely determined by the family of neighbourhoods of each of its points.

Proof.
$\Rightarrow$ Let (X, τ) be a topological space, $x \in X$ and $\mathcal{F}(x)$ the filter of neighbourhoods of x. Then (N1) trivially holds by definition of neighbourhood of x. To show (N2), let us take $A \in \mathcal{F}(x)$. Since A is a neighbourhood of x, there exists $B \in \tau$ s.t. $x \in B \subseteq A$. Then clearly $B \in \mathcal{F}(x)$. Moreover, since for any $y \in B$ we have that $y \in B \subseteq A$ and B is open, we can conclude that $A \in \mathcal{F}(y)$.
$\Leftarrow$ Assume that for any $x \in X$ we have a filter $\mathcal{F}_x$ fulfilling (N1) and (N2). Let us define $\tau := \{O \subseteq X : \text{if } x \in O \text{ then } O \in \mathcal{F}_x\}$. Since each $\mathcal{F}_x$ is a filter, τ is a topology. Indeed:

- $\emptyset \in \tau$ by definition of τ. Also $X \in \tau$, because for any $x \in X$ and any $A \in \mathcal{F}_x$ we clearly have $X \supseteq A$ and so by (F3) $X \in \mathcal{F}_x$.
- By (F2) we have that τ is closed under finite intersection.
- Let U be an arbitrary union of sets $U_i \in \tau$ and let $x \in U$. Then there exists at least one i s.t. $x \in U_i$ and so $U_i \in \mathcal{F}_x$ because $U_i \in \tau$. But $U \supseteq U_i$, then by (F3) we get that $U \in \mathcal{F}_x$ and so $U \in \tau$.

It remains to show that τ on X is actually s.t. $\mathcal{F}_x \equiv \mathcal{F}(x)$, $\forall x \in X$.

- Any $U \in \mathcal{F}(x)$ is a neighbourhood of x and so there exists $O \in \tau$ s.t. $x \in O \subseteq U$. Then, by definition of τ, we have $O \in \mathcal{F}_x$ and so (F3) implies that $U \in \mathcal{F}_x$. Hence, $\mathcal{F}(x) \subseteq \mathcal{F}_x$.
- Let $U \in \mathcal{F}_x$ and set $W := \{y \in U : U \in \mathcal{F}_y\} \subseteq U$. Since $x \in U$ by (N1), we also have $x \in W$. Moreover, if $y \in W$ then by (N2) there exists $V \in \mathcal{F}_y$ s.t. $\forall z \in V$ we have $U \in \mathcal{F}_z$. This means that $z \in W$ and so $V \subseteq W$. Then $W \in \mathcal{F}_y$ by (F3). Hence, we have showed that if $y \in W$ then $W \in \mathcal{F}_y$, i.e. $W \in \tau$. Summing up, we have just constructed an open set W s.t. $x \in W \subseteq U$, i.e. $U \in \mathcal{F}(x)$, and so $\mathcal{F}_x \subseteq \mathcal{F}(x)$.

Note that the non-empty open subsets of any other topology τ' on X such that $\mathcal{F}_x \equiv \mathcal{F}(x)$, $\forall x \in X$ must be identical to the subsets O of X for which $O \in \mathcal{F}_x$ whenever $x \in O$. Hence, $\tau' \equiv \tau$. □

Remark 1.1.10. *The previous proof in particular shows that a subset is open if and only if it contains a neighbourhood of each of its points.*

Definition 1.1.11. *Given a topological space X, a basis $\mathcal{B}(x)$ of the filter of neighbourhoods $\mathcal{F}(x)$ of $x \in X$ is called a* base of neighbourhoods of x*, i.e.$\mathcal{B}(x)$ is a subset of $\mathcal{F}(x)$ s.t. every set in $\mathcal{F}(x)$ contains one in $\mathcal{B}(x)$. The elements of $\mathcal{B}(x)$ are called* basic neighbourhoods of x*. If a base of neighbourhoods is given for any $x \in X$, we speak of* base of neighbourhoods of X*.*

Example 1.1.12. *The open sets of a topological space other than the empty set always form a base of neighbourhoods.*

Theorem 1.1.13. *Given a topological space X and a point $x \in X$, a base of open neighbourhoods $\mathcal{B}(x)$ satisfies the following properties.*

(B1) *For any $U \in \mathcal{B}(x)$, $x \in U$.*

(B2) *For any $U_1, U_2 \in \mathcal{B}(x)$, $\exists\, U_3 \in \mathcal{B}(x)$ s.t. $U_3 \subseteq U_1 \cap U_2$.*

(B3) *If $y \in U \in \mathcal{B}(x)$, then $\exists\, W \in \mathcal{B}(y)$ s.t. $W \subseteq U$.*

Viceversa, if for each point x in a set X we are given a collection of subsets $\mathcal{B}_x$ fulfilling the properties (B1), (B2) and (B3) then there exists a unique topology τ s.t. for each $x \in X$, $\mathcal{B}_x$ is a base of neighbourhoods of x, i.e. $\mathcal{B}_x \equiv \mathcal{B}(x)$, $\forall x \in X$.

Proof. The proof easily follows by using Theorem 1.1.9. □

The previous theorem gives a further way of introducing a topology on a set. Indeed, starting from a base of neighbourhoods of X, we can define a topology on X by setting that a set is open iff whenever it contains a point it also contains a basic neighbourhood of the point. Thus a topology on a set X is uniquely determined by a base of neighbourhoods of each of its points.

1.1.2 Comparison of topologies

Any set X may carry several different topologies. When we deal with topological vector spaces, we will very often encounter this situation of a set, in fact a vector space, carrying several topologies (all compatible with the linear structure, in a sense that is going to be specified soon). In this case, it is convenient being able to compare topologies.

Definition 1.1.14. *Let τ, τ' be two topologies on the same set X. We say that τ is* coarser *(or* weaker*) than τ', in symbols $\tau \subseteq \tau'$, if every subset of X which is open for τ is also open for τ', or equivalently, if every neighborhood of a point in X w.r.t. τ is also a neighborhood of that same point in the topology τ'. In this case τ' is said to be* finer *(or* stronger*) than τ'.*

Denote by $\mathcal{F}(x)$ and $\mathcal{F}'(x)$ the filter of neighbourhoods of a point $x \in X$ w.r.t. τ and w.r.t. τ', respectively. Then: τ is coarser than τ' iff for any point $x \in X$ we have $\mathcal{F}(x) \subseteq \mathcal{F}'(x)$ (this means that every subset of X which belongs to $\mathcal{F}(x)$ also belongs to $\mathcal{F}'(x)$).

Two topologies τ and τ' on the same set X *coincide* when they give the same open sets or the same closed sets or the same neighbourhoods of each point; equivalently, when τ is both coarser and finer than τ'. Two basis of neighbourhoods of a set are *equivalent* when they define the same topology.

Remark 1.1.15. *Given two topologies on the same set, it may very well happen that none is finer than the other. If it is possible to establish which one is finer, then we say that the two topologies are* comparable.

Example 1.1.16.
The cofinite topology τ_c *on* $\mathbb{R}$, *i.e.* $\tau_c := \{U \subseteq \mathbb{R} : U = \emptyset \text{ or } \mathbb{R} \setminus U \text{ is finite}\}$, *and the topology* τ_i *having* $\{(-\infty, a) : a \in \mathbb{R}\}$ *as a basis are incomparable. In fact, it is easy to see that* $\tau_i = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ *as these are the unions of sets in the given basis. In particular, we have that* $\mathbb{R} - \{0\}$ *is in* τ_c *but not* τ_i. *Moreover, we have that* $(-\infty, 0)$ *is in* τ_i *but not* τ_c. *Hence,* τ_c *and* τ_i *are incomparable.*

It is always possible to construct at least two topologies on every set X by choosing the collection of open sets to be as large or as small as possible:

- the *trivial topology*: every point of X has only one neighbourhood which is X itself. Equivalently, the only open subsets are $\emptyset$ and X. The only possible basis for the trivial topology is $\{X\}$.
- the *discrete topology*: given any point $x \in X$, every subset of X containing x is a neighbourhood of x. Equivalently, every subset of X is open (actually clopen). In particular, the singleton $\{x\}$ is a neighbourhood of x and actually is a basis of neighbourhoods of x. The collection of all singletons is a basis for the discrete topology.

The discrete topology on a set X is finer than any other topology on X, while the trivial topology is coarser than all the others. Topologies on a set form thus a partially ordered set, having a maximal and a minimal element, respectively the discrete and the trivial topology.

A useful criterion to compare topologies on the same set is the following:

Theorem 1.1.17 (Hausdorff's criterion)**.**
For each $x \in X$, *let* $\mathcal{B}(x)$ *a base of neighbourhoods of* x *for a topology* τ *on* X *and* $\mathcal{B}'(x)$ *a base of neighbourhoods of* x *for a topology* τ' *on* X.
$\tau \subseteq \tau'$ *iff* $\forall\, x \in X, \forall\, U \in \mathcal{B}(x)\ \exists\, V \in \mathcal{B}'(x)$ *s.t.* $x \in V \subseteq U$.