Proof. Let us first show that the collection $\mathcal{B}$ is a basis of neighbourhoods of the origin for the unique topology τ making X into a locally convex t.v.s. by using Theorem 4.1.14 and then let us prove that τ actually coincides with the topology induced by the family $\mathcal{P}$.

For any $i \in I$ and any $\varepsilon > 0$, consider the set $\{x \in X : p_i(x) < \varepsilon\} = \varepsilon \mathring{U}_{p_i}$. This is absorbing and absolutely convex, since we have already showed above that $\mathring{U}_{p_i}$ fulfills such properties. Therefore, any element of $\mathcal{B}$ is an absorbing absolutely convex subset of X as finite intersection of absorbing absolutely convex sets. Moreover, both properties a) and b) of Theorem 4.1.14 are clearly satisfied by $\mathcal{B}$. Hence, Theorem 4.1.14 guarantees that there exists a unique topology τ on X s.t. (X, τ) is a locally convex t.v.s. and $\mathcal{B}$ is a basis of neighbourhoods of the origin for τ.

Let us consider (X, τ). Then for any $i \in I$, the seminorm p_i is continuous, because for any $\varepsilon > 0$ we have $p_i^{-1}([0, \varepsilon[) = \{x \in X : p_i(x) < \varepsilon\} \in \mathcal{B}$ which means that $p_i^{-1}([0, \varepsilon[)$ is a neighbourhood of the origin in (X, τ). Therefore, the topology $\tau_{\mathcal{P}}$ induced by the family $\mathcal{P}$ is by definition coarser than τ. On the other hand, each p_i is also continuous w.r.t. $\tau_{\mathcal{P}}$ and so $\mathcal{B} \subseteq \tau_{\mathcal{P}}$. But $\mathcal{B}$ is a basis for τ, then necessarily τ is coarser than $\tau_{\mathcal{P}}$. Hence, $\tau \equiv \tau_{\mathcal{P}}$.

Viceversa, let us assume that (X, τ) is a locally convex t.v.s.. Then by Theorem 4.1.14 there exists a basis $\mathcal{N}$ of neighbourhoods of the origin in X consisting of absorbing absolutely convex sets s.t. the properties a) and b) in Theorem 4.1.14 are fulfilled. W.l.o.g. we can assume that they are open. Consider now the family $\mathcal{S} := \{p_N : N \in \mathcal{N}\}$. By Lemma 4.2.7, we know that each p_N is a seminorm and that $\mathring{U}_{p_N} \subseteq N$. Let us show that for any $N \in \mathcal{N}$ we have actually that $N = \mathring{U}_{p_N}$. Since any $N \in \mathcal{N}$ is open and the scalar multiplication is continuous we have that for any $x \in N$ there exists $0 < t < 1$ s.t. $x \in tN$ and so $p_N(x) \leq t < 1$, i.e. $x \in \mathring{U}_{p_N}$.

We want to show that the topology $\tau_{\mathcal{S}}$ induced by the family $\mathcal{S}$ coincides with original topology τ on X. We know from the first part of the proof how to construct a basis for a topology induced by a family of seminorms. In fact, a basis of neighbourhoods of the origin for $\tau_{\mathcal{S}}$ is given by

$$\mathcal{B} := \left\{ \bigcap_{i=1}^{n} \{x \in X : p_{N_i}(x) < \varepsilon\} : N_1, \ldots, N_n \in \mathcal{N}, n \in \mathbb{N}, \varepsilon > 0, \epsilon \in \mathbb{R} \right\}.$$

For any $N \in \mathcal{N}$ we showed that $N = \mathring{U}_{p_N} \in \mathcal{B}$ so by Hausdorff criterion we get $\tau \subseteq \tau_{\mathcal{S}}$. Also for any $B \in \mathcal{B}$ we have $B = \cap_{i=1}^{n} \varepsilon \mathring{U}_{p_{N_i}} = \cap_{i=1}^{n} \varepsilon N_i$ for some $n \in \mathbb{N}$, $N_1, \ldots, N_n \in \mathcal{N}$ and $\varepsilon > 0$. Then property b) of Theorem 4.1.14 for $\mathcal{N}$ implies that for each $i = 1, \ldots, n$ there exists $V_i \in \mathcal{N}$ s.t. $V_i \subseteq \varepsilon N_i$ and so by the property a) of $\mathcal{N}$ we have that there exists $V \in \mathbb{N}$ s.t. $V \subseteq \cap_{i=1}^{n} V_i \subseteq B$. Hence, by Hausdorff criterion $\tau_{\mathcal{S}} \subseteq \tau$. □

This result justifies why several authors define a locally convex space to be a t.v.s whose topology is induced by a family of seminorms (which is now evidently equivalent to Definition 4.1.11)

In the previous proofs we have used some interesting properties of semiballs in a vector space. For convenience, we collect them here together with some further ones which we will repeatedly use in the following.

Proposition 4.2.10. *Let X be a vector space and p a seminorm on X. Then:*

a) $\mathring{U}_p$ *is absorbing and absolutely convex.*

b) $\forall\, r > 0$, $r\mathring{U}_p = \{x \in X : p(x) < r\} = \mathring{U}_{\frac{1}{r}p}$.

c) $\forall\, x \in X$, $x + \mathring{U}_p = \{y \in X : p(y - x) < 1\}$.

d) If q is also a seminorm on X then: $p \leq q$ if and only if $\mathring{U}_q \subseteq \mathring{U}_p$.

e) If $n \in \mathbb{N}$ and $s_1, \ldots, s_n$ are seminorms on X, then their maximum *s defined as $s(x) := \max_{i=1,\ldots,n} s_i(x)$, $\forall x \in X$ is also seminorm on X and $\mathring{U}_s = \bigcap_{i=1}^n \mathring{U}_{s_i}$.*

All the previous properties also hold for closed semballs.

Proof.

a) This was already proved as part of Lemma 4.2.7.

b) For any $r > 0$, we have
$$r\mathring{U}_p = \{rx \in X : p(x) < 1\} = \underbrace{\{y \in X : \frac{1}{r}p(y) < 1\}}_{\mathring{U}_{\frac{1}{r}p}} = \{y \in X : p(y) < r\}.$$

c) For any $x \in X$, we have
$$x + \mathring{U}_p = \{x + z \in X : p(z) < 1\} = \{y \in X : p(y - x) < 1\}.$$

d) Suppose that $p \leq q$ and take any $x \in \mathring{U}_q$. Then we have $q(x) < 1$ and so $p(x) \leq q(x) < 1$, i.e. $x \in \mathring{U}_p$. Viceversa, suppose that $\mathring{U}_q \subseteq \mathring{U}_p$ holds and take any $x \in X$. We have that either $q(x) > 0$ or $q(x) = 0$. In the first case, for any $0 < \varepsilon < 1$ we get that $q\big(\frac{\varepsilon x}{q(x)}\big) = \varepsilon < 1$. Then $\frac{\varepsilon x}{q(x)} \in \mathring{U}_q$ which implies by our assumption that $\frac{\varepsilon x}{q(x)} \in \mathring{U}_p$ i.e. $p\big(\frac{\varepsilon x}{q(x)}\big) < 1$. Hence, $\varepsilon p(x) < q(x)$ and so when $\varepsilon \to 1$ we get $p(x) \leq q(x)$. If instead we are in the second case that is when $q(x) = 0$, then we claim that also $p(x) = 0$. Indeed, if $p(x) > 0$ then $q\big(\frac{x}{p(x)}\big) = 0$ and so $\frac{x}{p(x)} \in \mathring{U}_q$ which implies by our assumption that $\frac{x}{p(x)} \in \mathring{U}_p$, i.e. $p(x) < p(x)$ which is a contradiction.

e) It is easy to check, using basic properties of the maximum, that the subadditivity and the positive homogeneity of each s_i imply the same properties for s. In fact, for any $x, y \in X$ and for any $\lambda \in \mathbb{K}$ we get:

- $s(x+y) = \max_{i=1,\dots,n} s_i(x+y) \leq \max_{i=1,\dots,n} (s_i(x)+s_i(y))$
  $\leq \max_{i=1,\dots,n} s_i(x) + \max_{i=1,\dots,n} s_i(y) = s(x)+s(y)$
- $s(\lambda x) = \max_{i=1,\dots,n} s_i(\lambda x) = |\lambda| \max_{i=1,\dots,n} s_i(x) = |\lambda| s(x)$.

Moreover, if $x \in \mathring{U}_s$ then $\max_{i=1,\dots,n} s_i(x) < 1$ and so for all $i = 1,\dots,n$ we have $s_i(x) < 1$, i.e. $x \in \bigcap_{i=1}^n \mathring{U}_{s_i}$. Conversely, if $x \in \bigcap_{i=1}^n \mathring{U}_{s_i}$ then for all $i = 1,\dots,n$ we have $s_i(x) < 1$. Since $s(x)$ is the maximum over a finite number of terms, it will be equal to $s_j(x)$ for some $j \in \{1,\dots,n\}$ and therefore $s(x) = s_j(x) < 1$, i.e. $x \in \mathring{U}_s$.

□

Proposition 4.2.11. *Let X be a t.v.s. and p a seminorm on X. Then the following conditions are equivalent:*
a) the open unit semiball $\mathring{U}_p$ of p is an open set.
b) p is continuous at the origin.
c) the closed unit semiball U_p of p is a barrel neighbourhood of the origin.
d) p is continuous at every point.

Proof.
a) ⇒ *b)* Suppose that $\mathring{U}_p$ is open in the topology on X. Then for any $\varepsilon > 0$ we have that $p^{-1}([0,\varepsilon[) = \{x \in X : p(x) < \varepsilon\} = \varepsilon \mathring{U}_p$ is an open neighbourhood of the origin in X. This is enough to conclude that $p : X \to \mathbb{R}^+$ is continuous at the origin.

b) ⇒ *c)* Suppose that p is continuous at the origin, then $U_p = p^{-1}([0,1])$ is a closed neighbourhood of the origin. Since U_p is also absorbing and absolutely convex by Proposition 4.2.10-a), U_p is a barrel.

c) ⇒ *d)* Assume that c) holds and fix $o \neq x \in X$. Using Proposition 4.2.10 and Proposition 4.2.3, we get that for any $\varepsilon > 0$: $p^{-1}([-\varepsilon + p(x), p(x)+\varepsilon]) = \{y \in X : |p(y) - p(x)| \leq \varepsilon\} \supseteq \{y \in X : p(y-x) \leq \varepsilon\} = x + \varepsilon U_p$, which is a closed neighbourhood of x since X is a t.v.s. and by the assumption *c*). Hence, p is continuous at x.

d) ⇒ *a)* If p is continuous on X then a) holds because the preimage of an open set under a continuous function is open and $\mathring{U}_p = p^{-1}([0,1[)$. □

With such properties in our hands we are able to give a criterion to compare two locally convex topologies using their generating families of seminorms.

Theorem 4.2.12 (Comparison of l.c. topologies).
Let $\mathcal{P} = \{p_i\}_{i\in I}$ and $\mathcal{Q} = \{q_j\}_{j\in J}$ be two families of seminorms on the vector space X inducing respectively the topologies $\tau_\mathcal{P}$ and $\tau_\mathcal{Q}$, which both make X into a locally convex t.v.s.. Then $\tau_\mathcal{P}$ is finer than $\tau_\mathcal{Q}$ (i.e. $\tau_\mathcal{Q} \subseteq \tau_\mathcal{P}$) iff

$$\forall q \in \mathcal{Q}\ \exists n \in \mathbb{N},\, i_1, \dots, i_n \in I,\, C > 0 \ \text{ s.t. } \ Cq(x) \leq \max_{k=1,\dots,n} p_{i_k}(x),\ \forall x \in X. \tag{4.2}$$

Proof.
Let us first recall that, by Theorem 4.2.9, we have that

$$\mathcal{B}_\mathcal{P} := \left\{ \bigcap_{k=1}^{n} \varepsilon \mathring{U}_{p_{i_k}} : i_1, \dots, i_n \in I, n \in \mathbb{N}, \varepsilon > 0, \varepsilon \in \mathbb{R} \right\}$$

and

$$\mathcal{B}_\mathcal{Q} := \left\{ \bigcap_{k=1}^{n} \varepsilon \mathring{U}_{q_{j_k}} : j_1, \dots, j_n \in J, n \in \mathbb{N}, \varepsilon > 0, \varepsilon \in \mathbb{R} \right\}.$$

are respectively bases of neighbourhoods of the origin for $\tau_\mathcal{P}$ and $\tau_\mathcal{Q}$.

By using Proposition 4.2.10, the condition (4.2) can be rewritten as

$$\forall q \in \mathcal{Q},\ \exists n \in \mathbb{N},\, i_1, \dots, i_n \in I,\, C > 0 \ \text{ s.t. } \ C \bigcap_{k=1}^{n} \mathring{U}_{p_{i_k}} \subseteq \mathring{U}_q.$$

which means that

$$\forall q \in \mathcal{Q},\ \exists\ B_q \in \mathcal{B}_\mathcal{P} \ \text{ s.t. } \ B_q \subseteq \mathring{U}_q. \tag{4.3}$$

since $C \bigcap_{k=1}^{n} \mathring{U}_{p_{i_k}} \in \mathcal{B}_\mathcal{P}$.

Condition (4.3) means that for any $q \in \mathcal{Q}$ the set $\mathring{U}_q \in \tau_\mathcal{P}$, which by Proposition 4.2.11 is equivalent to say that q is continuous w.r.t. $\tau_\mathcal{P}$. By definition of $\tau_\mathcal{Q}$, this gives that $\tau_\mathcal{Q} \subseteq \tau_\mathcal{P}$. [1]

□

This theorem allows us to easily see that the topology induced by a family of seminorms on a vector space does not change if we close the family under taking the maximum of finitely many of its elements. Indeed, the following result holds.

[1] **Alternate proof without using Prop 4.2.11.** (Sheet 9, Exercise 1 a))

Proposition 4.2.13. *Let $\mathcal{P} := \{p_i\}_{i\in I}$ be a family of seminorms on a vector space X and $\mathcal{Q} := \left\{ \max_{i\in B} p_i : \emptyset \neq B \subseteq I \text{ with } B \text{ finite} \right\}$. Then $\mathcal{Q}$ is a family of seminorms and $\tau_{\mathcal{P}} = \tau_{\mathcal{Q}}$, where $\tau_{\mathcal{P}}$ and $\tau_{\mathcal{Q}}$ denote the topology induced on X by $\mathcal{P}$ and $\mathcal{Q}$, respectively.*

Proof.
First of all let us note that, by Proposition 4.2.10, $\mathcal{Q}$ is a family of seminorms. On the one hand, since $\mathcal{P} \subseteq \mathcal{Q}$, by definition of induced topology we have $\tau_{\mathcal{P}} \subseteq \tau_{\mathcal{Q}}$. On the other hand, for any $q \in \mathcal{Q}$ we have $q = \max_{i\in B} p_i$ for some $\emptyset \neq B \subseteq I$ finite. Then (4.2) is fulfilled for $n = |B|$ (where $|B|$ denotes the cardinality of the finite set B), $i_1, \ldots, i_n$ being the n elements of B and for any $0 < C \leq 1$. Hence, by Theorem 4.2.12, $\tau_{\mathcal{Q}} \subseteq \tau_{\mathcal{P}}$. □

This fact can be used to show the following very useful property of locally convex t.v.s.

Proposition 4.2.14. *The topology of a locally convex t.v.s. can be always induced by a* directed *family of seminorms.*

Definition 4.2.15. *A family $\mathcal{Q} := \{q_j\}_{j\in J}$ of seminorms on a vector space X is said to be* directed *if*

$$\forall\, j_1, j_2 \in J,\ \exists\, j \in J, C > 0 \ \text{ s.t. } \ Cq_j(x) \geq \max\{q_{j_1}(x), q_{j_2}(x)\},\ \forall\, x \in X \quad (4.4)$$

or equivalently by induction if

$$\forall\ n \in \mathbb{N}, j_1, \ldots, j_n \in J,\ \exists\, j \in J, C > 0 \ \text{ s.t. } \ Cq_j(x) \geq \max_{k=1,\ldots,n} q_{j_k}(x),\ \forall\, x \in X.$$

Proof. of Proposition 4.2.14
Let (X, τ) be a locally convex t.v.s.. By Theorem 4.2.9, we have that there exists a family of seminorms $\mathcal{P} := \{p_i\}_{i\in I}$ on X s.t. $\tau = \tau_{\mathcal{P}}$. Let us define $\mathcal{Q}$ as the collection obtained by forming the maximum of finitely many elements of $\mathcal{P}$, i.e. $\mathcal{Q} := \left\{ \max_{i\in B} p_i : \emptyset \neq B \subseteq I \text{ with } B \text{ finite} \right\}$. By Proposition 4.2.13, $\mathcal{Q}$ is a family of seminorms and we have that $\tau_{\mathcal{P}} = \tau_{\mathcal{Q}}$. We claim that $\mathcal{Q}$ is directed.

Let $q, q' \in \mathcal{Q}$, i.e. $q := \max_{i\in B} p_i$ and $q' := \max_{i\in B'} p_i$ for some non-empty finite subsets B, B' of I. Let us define $q'' := \max_{i\in B\cup B'} p_i$. Then $q'' \in \mathcal{Q}$ and for any $C \geq 1$ we have that (4.4) is satisfied, because we get that for any $x \in X$

$$Cq''(x) = C \max\left\{ \max_{i\in B} p_i(x), \max_{i\in B'} p_i(x) \right\} \geq \max\{q(x), q'(x)\}.$$

Hence, $\mathcal{Q}$ is directed. □

It is possible to show (Sheet 9, Exercise 3) that a basis of neighbourhoods of the origin for the l.c. topology $\tau_{\mathcal{Q}}$ induced by a directed family of seminorms $\mathcal{Q}$ is given by:

$$\mathcal{B}_d := \{r\mathring{U}_q : q \in \mathcal{Q}, r > 0\}. \tag{4.5}$$

4.3 Hausdorff locally convex t.v.s

In Section 2.2, we gave some characterization of Hausdorff t.v.s. which can of course be applied to establish whether a locally convex t.v.s. is Hausdorff or not. However, in this section we aim to provide necessary and sufficient conditions bearing only on the family of seminorms generating a locally convex topology for being a Hausdorff topology.

Definition 4.3.1.
A family of seminorms $\mathcal{P} := \{p_i\}_{i\in I}$ on a vector space X is said to be separating *if*

$$\forall x \in X \setminus \{o\}, \exists\, i \in I \text{ s.t. } p_i(x) \neq 0. \tag{4.6}$$

Note that the separation condition (4.6) is equivalent to

$$p_i(x) = 0,\ \forall i \in I \ \Rightarrow x = o$$

which by using Proposition 4.2.10 can be rewritten as

$$\bigcap_{i\in I, c>0} c\mathring{U}_{p_i} = \{o\},$$

since $p_i(x) = 0$ is equivalent to say that $p_i(x) < c$, for all $c > 0$.

Lemma 4.3.2. *Let $\tau_{\mathcal{P}}$ be the topology induced by a separating family of seminorms $\mathcal{P} := (p_i)_{i\in I}$ on a vector space X. Then $\tau_{\mathcal{P}}$ is a Hausdorff topology.*

Proposition 4.3.3. *A locally convex t.v.s. is Hausdorff if and only if its topology can be induced by a separating family of seminorms.*

Examples 4.3.4.

1. *Every normed space is a Hausdorff locally convex space, since every norm is a seminorm satisfying the separation property. Therefore, every Banach space is a complete Hausdorff locally convex space.*
2. *Every family of seminorms on a vector space containing a norm induces a Hausdorff locally convex topology.*
3. *Given an open subset* Ω *of* $\mathbb{R}^d$ *with the euclidean topology, the space* $\mathcal{C}(\Omega)$ *of real valued continuous functions on* Ω *with the so-called* topology of uniform convergence on compact sets *is a locally convex t.v.s.. This topology is defined by the family* $\mathcal{P}$ *of all the seminorms on* $\mathcal{C}(\Omega)$ *given by*

$$p_K(f) := \max_{x \in K} |f(x)|, \forall K \subset \Omega \text{ compact }.$$

Moreover, $(\mathcal{C}(\Omega), \tau_{\mathcal{P}})$ *is Hausdorff, because the family* $\mathcal{P}$ *is clearly separating. In fact, if* $p_K(f) = 0$, $\forall K$ *compact subsets of* Ω *then in particular* $p_{\{x\}}(f) = |f(x)| = 0$ $\forall x \in \Omega$, *which implies* $f \equiv 0$ *on* Ω.

More generally, for any X *locally compact we have that* $\mathcal{C}(X)$ *with the topology of uniform convergence on compact subsets of* X *is a locally convex Hausdorff t.v.s.*