Hence, $\mathcal{Q}$ is directed. $\square$

It is possible to show (Sheet 9, Exercise 3) that a basis of neighbourhoods of the origin for the l.c. topology $\tau_{\mathcal{Q}}$ induced by a directed family of seminorms $\mathcal{Q}$ is given by:

$$\mathcal{B}_d := \{r\mathring{U}_q : q \in \mathcal{Q}, r > 0\}. \tag{4.5}$$

4.3 Hausdorff locally convex t.v.s

In Section 2.2, we gave some characterization of Hausdorff t.v.s. which can of course be applied to establish whether a locally convex t.v.s. is Hausdorff or not. However, in this section we aim to provide necessary and sufficient conditions bearing only on the family of seminorms generating a locally convex topology for being a Hausdorff topology.

Definition 4.3.1.
A family of seminorms $\mathcal{P} := \{p_i\}_{i\in I}$ on a vector space X is said to be separating *if*

$$\forall x \in X \setminus \{o\}, \exists\ i \in I \text{ s.t. } p_i(x) \neq 0. \tag{4.6}$$

Note that the separation condition (4.6) is equivalent to

$$p_i(x) = 0,\ \forall i \in I\ \Rightarrow x = o$$

which by using Proposition 4.2.10 can be rewritten as

$$\bigcap_{i\in I, c>0} c\mathring{U}_{p_i} = \{o\},$$

since $p_i(x) = 0$ is equivalent to say that $p_i(x) < c$, for all $c > 0$.

It is clear that if any of the elements in a family of seminorms is actually a norm, then the the family is separating.

Lemma 4.3.2. *Let $\tau_{\mathcal{P}}$ be the topology induced by a separating family of seminorms $\mathcal{P} := (p_i)_{i\in I}$ on a vector space X. Then $\tau_{\mathcal{P}}$ is a Hausdorff topology.*

Proof. Let $x, y \in X$ be such that $x \neq y$. Since $\mathcal{P}$ is separating, we have that $\exists\ i \in I$ with $p_i(x - y) \neq 0$. Then $\exists\ \epsilon > 0$ s.t. $p_i(x - y) = 2\epsilon$. Let us define $V_x := \{u \in X \mid p_i(x - u) < \epsilon\}$ and $V_y := \{u \in X \mid p_i(y - u) < \epsilon\}$. By

Proposition 4.2.10, we get that $V_x = x+\varepsilon \mathring{U}_{p_i}$ and $V_y = y+\varepsilon \mathring{U}_{p_i}$. Since Theorem 4.2.9 guarantees that $(X, \tau_{\mathcal{P}})$ is a t.v.s. where the set $\varepsilon \mathring{U}_{p_i}$ is a neighbourhood of the origin, V_x and V_y are neighbourhoods of x and y, respectively. They are clearly disjoint. Indeed, if there would exist $u \in V_x \cap V_y$ then

$$p_i(x-y) = p_i(x-u+u-y) \leq p_i(x-u) + p_i(u-y) < 2\varepsilon$$

which is a contradiction. □

Proposition 4.3.3. *A locally convex t.v.s. is Hausdorff if and only if its topology can be induced by a separating family of seminorms.*

Proof. Let (X, τ) be a locally convex t.v.s.. Then we know that there always exists a basis $\mathcal{N}$ of neighbourhoods of the origin in X consisting of open absorbing absolutely convex sets. Moreover, in Theorem 4.2.9, we have showed that $\tau = \tau_{\mathcal{P}}$ where $\mathcal{P}$ is the family of seminorms given by the Minkowski functionals of sets in $\mathcal{N}$, i.e. $\mathcal{P} := \{p_N : N \in \mathcal{N}\}$, and also that for each $N \in \mathcal{N}$ we have $N = \mathring{U}_{p_N}$.

Suppose that (X, τ) is also Hausdorff. Then Proposition 2.2.3 ensures that for any $x \in X$ with $x \neq o$ there exists a neighbourhood V of the origin in X s.t. $x \notin V$. This implies that there exists at least $N \in \mathcal{N}$ s.t. $x \notin N$ [2]. Hence, $x \notin N = \mathring{U}_{p_N}$ means that $p_N(x) \geq 1$ and so $p_N(x) \neq 0$, i.e. $\mathcal{P}$ is separating.

Conversely, if τ is induced by a separating family of seminorms $\mathcal{P}$, i.e. $\tau = \tau_{\mathcal{P}}$, then Lemma 4.3.2 ensures that X is Hausdorff.

□

Examples 4.3.4.

1. *Every normed space is a Hausdorff locally convex space, since every norm is a seminorm satisfying the separation property. Therefore, every Banach space is a complete Hausdorff locally convex space.*
2. *Every family of seminorms on a vector space containing a norm induces a Hausdorff locally convex topology.*
3. *Given an open subset Ω of $\mathbb{R}^d$ with the euclidean topology, the space $\mathcal{C}(\Omega)$ of real valued continuous functions on Ω with the so-called* topology of uniform convergence on compact sets *is a locally convex t.v.s.. This topology is defined by the family $\mathcal{P}$ of all the seminorms on $\mathcal{C}(\Omega)$ given by*
$$p_K(f) := \max_{x \in K} |f(x)|, \ \forall K \subset \Omega \text{ compact.}$$

[2] Since $\mathcal{N}$ is a basis of neighbourhoods of the origin, $\exists\, M \in \mathcal{N}$ s.t. $M \subseteq V$. If x would belong to all elements of the basis then in particular it would be $x \in M$ and so also $x \in V$, contradiction.

Moreover, $(\mathcal{C}(\Omega), \tau_{\mathcal{P}})$ is Hausdorff, because the family $\mathcal{P}$ is clearly separating. In fact, if $p_K(f) = 0$, $\forall K$ compact subsets of Ω then in particular $p_{\{x\}}(f) = |f(x)| = 0$ $\forall x \in \Omega$, which implies $f \equiv 0$ on Ω.

More generally, for any X locally compact we have that $\mathcal{C}(X)$ with the topology of uniform convergence on compact subsets of X is a locally convex Hausdorff t.v.s.

To introduce two other examples of l.c. Hausdorff t.v.s. we need to recall some standard general notations. Let $\mathbb{N}_0$ be the set of all non-negative integers. For any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ one defines $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. For any $\beta \in \mathbb{N}_0^d$, the symbol D^β denotes the partial derivative of order $|\beta|$ where $|\beta| := \sum_{i=1}^d \beta_i$, i.e.

$$D^\beta := \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}} = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_d}}{\partial x_d^{\beta_d}}.$$

Examples 4.3.5.

1. *Let $\Omega \subseteq \mathbb{R}^d$ open in the euclidean topology. For any $k \in \mathbb{N}_0$, let $\mathcal{C}^k(\Omega)$ be the set of all real valued $k-$*times continuously differentiable functions *on Ω, i.e. all the derivatives of f of order $\leq k$ exist (at every point of Ω) and are continuous functions in Ω. Clearly, when $k = 0$ we get the set $\mathcal{C}(\Omega)$ of all real valued continuous functions on Ω and when $k = \infty$ we get the so-called set of all* infinitely differentiable functions *or* smooth functions *on Ω. For any $k \in \mathbb{N}_0$, $\mathcal{C}^k(\Omega)$ (with pointwise addition and scalar multiplication) is a vector space over $\mathbb{R}$. The topology given by the following family of seminorms on $\mathcal{C}^k(\Omega)$:*

$$p_{m,K}(f) := \sup_{\substack{\beta \in \mathbb{N}_0^d \\ |\beta| \leq m}} \sup_{x \in K} \left|(D^\beta f)(x)\right|, \forall K \subseteq \Omega \text{ compact}, \forall m \in \{0, 1, \dots, k\},$$

   makes $\mathcal{C}^k(\Omega)$ into a l.c. Hausdorff t.v.s. (see Sheet 9, Exercise 2-a) for the proof in the case $k = \infty$).
2. *The* Schwartz space *or* space of rapidly decreasing functions *on $\mathbb{R}^d$ is defined as the set $\mathcal{S}(\mathbb{R}^d)$ of all real-valued functions which are defined and infinitely differentiable on $\mathbb{R}^d$ and which have the additional property (regulating their growth at infinity) that all their derivatives tend to zero at infinity faster than any inverse power of x, i.e.*

$$\mathcal{S}(\mathbb{R}^d) := \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} \left| x^\alpha D^\beta f(x) \right| < \infty, \ \forall \alpha, \beta \in \mathbb{N}_0^d \right\}.$$

(For example, any smooth function f with compact support in $\mathbb{R}^d$ is in $\mathcal{S}(\mathbb{R}^d)$, since any derivative of f is continuous and supported on a compact subset of $\mathbb{R}^d$, so $x^\alpha(D^\beta f(x))$ has a maximum in $\mathbb{R}^d$ by the extreme value theorem.)

The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is a vector space over $\mathbb{R}$ and the topology given by the family $\mathcal{Q}$ of seminorms on $\mathcal{S}(\mathbb{R}^d)$:

$$q_{\alpha,\beta}(f) := \sup_{x\in\mathbb{R}^d} \left| x^\alpha D^\beta f(x) \right|, \ \ \forall \alpha, \beta \in \mathbb{N}_0^d$$

makes $\mathcal{S}(\mathbb{R}^d)$ into a l.c. Hausdorff t.v.s. (see Sheet 9, Exercise 2-b)). Note that $\mathcal{S}(\mathbb{R}^d)$ is a linear subspace of $\mathcal{C}^\infty(\mathbb{R}^d)$, but its topology $\tau_\mathcal{Q}$ on $\mathcal{S}(\mathbb{R}^d)$ is finer than the subspace topology induced on it by $\mathcal{C}^\infty(\mathbb{R}^d)$ (see Sheet 9, Exercise 2-c)).

4.4 The finest locally convex topology

In the previous sections we have seen how to generate topologies on a vector space which makes it into a locally convex t.v.s.. Among all of them, there is the finest one (i.e. the one having the largest number of open sets) which is usually called the *finest locally convex topology* on the given vector space.

Proposition 4.4.1. *The finest locally convex topology on a vector space X is the topology induced by the family of all seminorms on X and it is a Hausdorff topology.*

Proof.
Let us denote by $\mathcal{S}$ the family of all seminorms on the vector space X. By Theorem 4.2.9, we know that the topology $\tau_\mathcal{S}$ induced by $\mathcal{S}$ makes X into a locally convex t.v.s. We claim that $\tau_\mathcal{S}$ is the finest locally convex topology. In fact, if there was a finer locally convex topology τ (i.e. if $\tau_\mathcal{S} \subseteq \tau$ with (X, τ) locally convex t.v.s.) then Theorem 4.2.9 would give that τ is also induced by a family $\mathcal{P}$ of seminorms. But surely $\mathcal{P} \subseteq \mathcal{S}$ and so $\tau = \tau_\mathcal{P} \subseteq \tau_\mathcal{S}$ by definition of induced topology. Hence, $\tau = \tau_\mathcal{S}$.

It remains to show that $(X, \tau_\mathcal{S})$ is Hausdorff. By Lemma 4.3.2, it is enough to prove that $\mathcal{S}$ is separating. Let $x \in X \setminus \{o\}$ and let $\mathcal{B}$ be an algebraic basis of the vector space X containing x (its existence is guaranteed by Zorn's lemma). Define the linear functional $L : X \to \mathbb{K}$ as $L(x) = 1$ and $L(y) = 0$

for all $y \in \mathcal{B} \setminus \{x\}$. Then it is easy to see that $s := |L|$ is a seminorm, so $s \in \mathcal{S}$ and $s(x) \neq 0$, which proves that $\mathcal{S}$ is separating.[3] □

An alternative way of describing the finest locally convex topology on a vector space X without using the seminorms is the following:

Proposition 4.4.2. *The collection of all absorbing absolutely convex sets of a vector space X is a basis of neighbourhoods of the origin for the finest locally convex topology on X.*

Proof. Let τ_{max} be the finest locally convex topology on X and $\mathcal{A}$ the collection of all absorbing absolutely convex sets of X. Since $\mathcal{A}$ fulfills all the properties required in Theorem 4.1.14, there exists a unique topology τ which makes X into a locally convex t.v.s.. Hence, by definition of finest locally convex topology, $\tau \subseteq \tau_{max}$. On the other hand, (X, τ_{max}) is itself locally convex and so Theorem 4.1.14 ensures that has a basis $\mathcal{B}_{max}$ of neighbourhoods of the origin consisting of absorbing absolutely convex subsets of X. Then clearly $\mathcal{B}_{max}$ is contained in $\mathcal{A}$ and, hence, $\tau_{max} \subseteq \tau$. □

This result can be proved also using Proposition 4.4.1 and the correspondence between Minkowski functionals and absorbing absolutely convex subsets of X introduced in the Section 4.2 (Sheet 10, Exercise 4).

Proposition 4.4.3. *Every linear functional on a vector space X is continuous w.r.t. the finest locally convex topology on X.*

Proof. Let $L : X \to \mathbb{K}$ be a linear functional on a vector space X. For any $\varepsilon > 0$, we denote by $B_\varepsilon(0)$ the open ball in $\mathbb{K}$ of radius ε and center $0 \in \mathbb{K}$, i.e. $B_\varepsilon(0) := \{k \in \mathbb{K} : |k| < \varepsilon\}$. Then we have that $L^{-1}(B_\varepsilon(0)) = \{x \in X : |L(x)| < \varepsilon\}$. It is easy to verify that the latter is an absorbing absolutely convex subset of X and so, by Proposition 4.4.2, it is a neighbourhood of the origin in the finest locally convex topology on X. Hence L is continuous at the origin and so, by Proposition 2.1.15-3), L is continuous everywhere in X. □

[3] Alternatively, we can show that $\mathcal{S}$ is separating by proving that there always exists a norm on X. In fact, let $\mathcal{B} = (b_i)_{i \in I}$ be an algebraic basis of X then for any $x \in X$ there exist a finite subset J of I and $\lambda_j \in \mathbb{K}$ for all $j \in J$ s.t. $x = \sum_{j \in J} \lambda_j b_j$ and so we can define $\|x\| := \max_{j \in J} |\lambda_j|$. Then it is easy to check that $\|\cdot\|$ is a norm on X. Hence, $\mathcal{S}$ always contains the norm $\|\cdot\|$ and so it is separating.

4.5 Finite topology on a countable dimensional t.v.s.

In this section we are going to give an important example of finest locally convex topology on an infinite dimensional vector space, namely the *finite topology* on any countable dimensional vector space. For simplicity, we are going to focus on $\mathbb{R}-$vector spaces.

Definition 4.5.1. *Let X be an infinite dimensional vector space whose dimension is countable. The* finite topology τ_f *on X is defined as follows:*
$U \subseteq X$ is open in τ_f iff $U \cap W$ is open in the euclidean topology on W for all finite dimensional subspaces W of X.
Equivalently, if we fix a Hamel basis $\{x_n\}_{n\in\mathbb{N}}$ of X and if for any $n \in \mathbb{N}$ we set $X_n := span\{x_1, \ldots, x_n\}$ s.t. $X = \bigcup_{i=1}^{\infty} X_i$ and $X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots$, then $U \subseteq X$ is open in τ_f iff $U \cap X_i$ is open in the euclidean topology on X_i for every $i \in \mathbb{N}$.

We actually already know a concrete example of countable dimensional space with the finite topology:

Example 4.5.2. *Let $n \in \mathbb{N}$ and $\underline{x} = (x_1, \ldots, x_n)$. Denote by $\mathbb{R}[\underline{x}]$ the space of polynomials in the n variables $x_1, \ldots, x_n$ with real coefficients and by*

$$\mathbb{R}_d[\underline{x}] := \{f \in \mathbb{R}[\underline{x}] \,|\, \deg f \leq d\}, d \in \mathbb{N}_0,$$

then $\mathbb{R}[\underline{x}] := \bigcup_{d=0}^{\infty} \mathbb{R}_d[\underline{x}]$. The finite topology τ_f on $\mathbb{R}[\underline{x}]$ is then given by:
$U \subseteq \mathbb{R}[\underline{x}]$ is open in τ_f iff $\forall d \in \mathbb{N}_0, U \cap \mathbb{R}_d[\underline{x}]$ is open in $\mathbb{R}_d[\underline{x}]$ with the euclidean topology.

Theorem 4.5.3. *Let X be an infinite dimensional vector space whose dimension is countable endowed with the finite topology τ_f. Then:*
a) (X, τ_f) is a Hausdorff locally convex t.v.s.
b) τ_f is the finest locally convex topology on X