4.5 Finite topology on a countable dimensional t.v.s.

In this section we are going to give an important example of finest locally convex topology on an infinite dimensional vector space, namely the *finite topology* on any countable dimensional vector space. For simplicity, we are going to focus on $\mathbb{R}-$vector spaces.

Definition 4.5.1. *Let X be an infinite dimensional vector space whose dimension is countable. The* finite topology *τ_f on X is defined as follows:*
$U \subseteq X$ is open in τ_f iff $U \cap W$ is open in the euclidean topology on W for all finite dimensional subspaces W of X.
Equivalently, if we fix a Hamel basis $\{x_n\}_{n\in\mathbb{N}}$ of X and if for any $n \in \mathbb{N}$ we set $X_n := span\{x_1, \ldots, x_n\}$ s.t. $X = \bigcup_{i=1}^{\infty} X_i$ and $X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots$, then $U \subseteq X$ is open in τ_f iff $U \cap X_i$ is open in the euclidean topology on X_i for every $i \in \mathbb{N}$.

We actually already know a concrete example of countable dimensional space with the finite topology:

Example 4.5.2. *Let $n \in \mathbb{N}$ and $\underline{x} = (x_1, \ldots, x_n)$. Denote by $\mathbb{R}[\underline{x}]$ the space of polynomials in the n variables $x_1, \ldots, x_n$ with real coefficients and by*

$$\mathbb{R}_d[\underline{x}] := \{f \in \mathbb{R}[\underline{x}] | \deg f \leq d\}, d \in \mathbb{N}_0,$$

then $\mathbb{R}[\underline{x}] := \bigcup_{d=0}^{\infty} \mathbb{R}_d[\underline{x}]$. The finite topology τ_f on $\mathbb{R}[\underline{x}]$ is then given by:
$U \subseteq \mathbb{R}[\underline{x}]$ is open in τ_f iff $\forall d \in \mathbb{N}_0, U \cap \mathbb{R}_d[\underline{x}]$ is open in $\mathbb{R}_d[\underline{x}]$ with the euclidean topology.

Theorem 4.5.3. *Let X be an infinite dimensional vector space whose dimension is countable endowed with the finite topology τ_f. Then:*
a) (X, τ_f) is a Hausdorff locally convex t.v.s.
b) τ_f is the finest locally convex topology on X

Proof.
a) We leave to the reader the proof of the fact that τ_f is compatible with the linear structure of X (Sheet 10, Exercise 3) and we focus instead on proving that τ_f is a locally convex topology. To this aim we are going to show that for any open neighbourhood U of the origin in (X, τ_f) there exists an open convex neighbourhood $U' \subseteq U$.

Let $\{x_i\}_{i\in\mathbb{N}}$ be an $\mathbb{R}$-basis for X and set $X_n := span\{x_1, \ldots, x_n\}$ for any $n \in \mathbb{N}$. We proceed (by induction on $n \in \mathbb{N}$) to construct an increasing sequence $C_n \subseteq U \cap X_n$ of convex subsets as follows:

- For $n=1$: Since $U\cap X_1$ is open in $X_1=\mathbb{R}x_1$, we have that there exists $a_1\in\mathbb{R}_+$ such that $C_1:=\left\{\lambda_1x_1 \mid -a_1\leq\lambda_1\leq a_1\right\}\subseteq U\cap X_1$.
- Inductive assumption on n: We assume we have found $a_1,\ldots,a_n\in\mathbb{R}_+$ such that $C_n:=\left\{\lambda_1x_1+\ldots+\lambda_nx_n \mid -a_i\leq\lambda_i\leq a_i\,;i\in\{1,\ldots,n\}\right\}\subseteq U\cap X_n$. Note that C_n is closed in X_n as well as in X_{n+1}.
- For $n+1$: We claim $\exists\ a_{n+1}\in\mathbb{R}_+$ such that
  $C_{n+1}:=\left\{\lambda_1x_1+\ldots+\lambda_nx_n+\lambda_{n+1}x_{n+1}|-a_i\leq\lambda_i\leq a_i\,;i\in\{1,\ldots,n+1\}\right\}\subseteq U\cap X_{n+1}$.
  Proof of claim: If the claim does not hold, then $\forall\ N\in\mathbb{N}\ \exists\ x^N\in X_{n+1}$ s.t.
  $$x^N=\lambda_1^Nx_1+\ldots\lambda_n^Nx_n+\lambda_{n+1}^Nx_{n+1}$$
  with $-a_i\leq\lambda_i^N\leq a_i$ for $i\in\{1,\ldots,n\}$, $-\frac{1}{N}\leq\lambda_{n+1}^N\leq\frac{1}{N}$ and $x^N\notin U$.
  But x^N has form $x^N=\underbrace{\lambda_1^Nx_1+\ldots+\lambda_n^Nx_n}_{\in\, C_n}+\lambda_{n+1}^Nx_{n+1}$, so $\{x^N\}_{N\in\mathbb{N}}$ is a bounded sequence in $X_{n+1}\setminus U$. Therefore, we can find a subsequence $\{x^{N_j}\}_{j\in\mathbb{N}}$ which is convergent as $j\to\infty$ to $x\in C_n\subseteq U$ (since C_n is closed in X_{n+1} and the $(n+1)-$th component of x^{N_j} tends to 0 as $j\to\infty$). Hence, the sequence $\{x^{N_j}\}_{j\in\mathbb{N}}\subseteq X_{n+1}\setminus U$ converges to $x\in U$ but this contradicts the fact that $X_{n+1}\setminus U$ is closed in X_{n+1}. This establishes the claim.

Now for any $n\in\mathbb{N}$ consider
$$D_n:=\left\{\lambda_1x_1+\ldots+\lambda_nx_n \mid -a_i<\lambda_i<a_i\,;i\in\{1,\ldots,n\}\right\},$$
then $D_n\subset C_n\subseteq U\cap X_n$ is open and convex in X_n. Then $U':=\cup_{n\in\mathbb{N}}D_n$ is an open and convex neighbourhood of the origin in (X,τ_f) and $U'\subseteq U$.

b) Let us finally show that τ_f is actually the finest locally convex topology τ_{max} on X which gives in turn also that (X,τ_f) is Hausdorff. Since we have already showed that τ_f is a l.c. topology on X, clearly we have $\tau_f\subseteq\tau_{max}$ by definition of finest l.c. topology on X.

Conversely, let us consider $U\subseteq X$ open in τ_{max}. We want to show that U is open in τ_f, i.e. $W\cap U$ is open in the euclidean topology on W for any finite dimensional subspace W of X. Now each W inherits τ_{max} from X. Let us denote by τ_{max}^W the subspace topology induced by (X,τ_{max}) on W. By definition of subspace topology, we have that $W\cap U$ is open in τ_{max}^W. Moreover, by Proposition 4.4.1, we know that (X,τ_{max}) is a Hausdorff t.v.s. and so (W,τ_{max}^W) is a finite dimensional Hausdorff t.v.s. (see by Proposition 2.1.15-1). Therefore, τ_{max}^W has to coincide with the euclidean topology by Theorem 3.1.1 and, consequently, $W\cap U$ is open w.r.t. the euclidean topology on W. ☐

4.6 Continuity of linear mappings on locally convex spaces

Since locally convex spaces are a particular class of topological vector spaces, the natural functions to be considered on this spaces are continuous linear maps. In this section, we present a necessary and sufficient condition for the continuity of a linear map between two l.c. spaces, bearing only on the seminorms inducing the two topologies.

For simplicity, let us start with linear functionals on a l.c. space. Recall that for us $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ endowed with the euclidean topology given by the absolute value $|\cdot|$. In this section, for any $\varepsilon > 0$ we denote by $B_\varepsilon(0)$ the open ball in $\mathbb{K}$ of radius ε and center $0 \in \mathbb{K}$ i.e. $B_\varepsilon(0) := \{k \in \mathbb{K} : |k| < \varepsilon\}$.

Proposition 4.6.1. *Let τ be a locally convex topology on a vector space X generated by a directed family $\mathcal{Q}$ of seminorms on X. Then $L : X \to \mathbb{K}$ is a τ-continuous linear functional iff there exists $q \in \mathcal{Q}$ such that L is q-continuous, i.e.*

$$\exists q \in \mathcal{Q}, \exists C > 0 \ \ s.t. \, |L(x)| \leq Cq(x), \, \forall \, x \in X. \tag{4.7}$$

Proof.
Let us first observe that since X and $\mathbb{K}$ are both t.v.s. by Proposition 2.1.15-3) the continuity of L is equivalent to its continuity at the origin. Therefore, it is enough to prove the criterion for the continuity of L at the origin.

The τ-continuity of L at the origin in X means that for any $\varepsilon > 0$ $L^{-1}(B_\varepsilon(0)) = \{x \in X : |L(x)| < \varepsilon\}$ is an open neighbourhood of the origin in (X, τ). Since the family $\mathcal{Q}$ inducing τ is directed, a basis of neighbourhood of the origin in (X, τ) is given by $\mathcal{B}_d$ as in (4.5). Therefore, L is τ-continuous at the origin in X if and only if $\forall \, \varepsilon > 0$, $\exists B \in \mathcal{B}_d$ s.t. $B \subseteq L^{-1}(B_\varepsilon(0))$, i.e.

$$\forall \, \varepsilon > 0, \, \exists \, q \in \mathcal{Q}, \, \exists \, r > 0 \ \text{ s.t. } \ r\mathring{U}_q \subseteq L^{-1}(B_\varepsilon(0)). \tag{4.8}$$

[4] ($\Rightarrow$) Suppose L is τ-continuous at the origin in X then (4.8) implies that L is $q-$continuous at the origin, because $r\mathring{U}_q$ is clearly an open neighbourhood of the origin in X w.r.t. the topology generated by the single seminorm q.

($\Leftarrow$) Suppose that there exists $q \in \mathcal{Q}$ s.t. L is q-continuous in X. Then, since τ is the topology induced by the whole family $\mathcal{Q}$ which is finer than the one induced by the single seminorm q, we clearly have that L is also $\tau-$continuous.

□

[4] Alternative proof: By simply observing that $|L|$ is a seminorm and by using Proposition 4.2.10, one can get that (4.7) is equivalent to (4.8) and so to the q-continuity of L at the origin.

By using this result together with Proposition 4.2.14 we get the following.

Corollary 4.6.2. *Let τ be a locally convex topology on a vector space X generated by a family $\mathcal{P} := \{p_i\}_{i\in I}$ of seminorms on X. Then $L : X \to \mathbb{K}$ is a τ-continuous linear functional iff there exist $n \in \mathbb{N}$, $i_1, \dots, i_n \in I$ such that L is $\left(\max_{k=1,\dots,n} p_{i_k}\right)$-continuous, i.e.*

$$\exists\, n \in \mathbb{N}, \exists\, i_1, \dots, i_n \in I, \exists\, C > 0 \;\; s.t. \;\; |L(x)| \leq C \max_{k=1,\dots,n} p_{i_k}(x),\, \forall\, x \in X.$$

The proof of Proposition 4.6.1 can be easily modified to get the following more general criterion for the continuity of any linear map between two locally convex spaces.

Theorem 4.6.3. *Let X and Y be two locally convex t.v.s. whose topologies are respectively generated by the families $\mathcal{P}$ and $\mathcal{Q}$ of seminorms on X. Then $f : X \to Y$ linear is continuous iff*

$$\forall\, q \in \mathcal{Q}, \exists\, n \in \mathbb{N}, \exists\, p_1, \dots, p_n \in \mathcal{P}, \exists\, C > 0 \,:\, q(f(x)) \leq C \max_{i=1,\dots,n} p_i(x),\, \forall\, x \in X.$$

Proof. (Sheet 11, Exercise 2) □

Chapter 5

The Hahn-Banach Theorem and its applications

5.1 The Hahn-Banach Theorem

One of the most important results in the theory of t.v.s. is the Hahn-Banach theorem (HBT). It is named for Hans Hahn and Stefan Banach who proved this theorem independently in the late 1920s, dealing with the problem of extending continuous linear functionals defined on a subspace of a seminormed vector space to the whole space. We will see that actually this extension problem can be reduced to the problem of separating by a closed hyperplane a convex open set and an affine submanifold (the image by a translation of a linear subspace) which do not intersect. Indeed, there are several versions of HBT in literature, but we are going to present just two of them as representatives of the analytic and the geometric side of this result.

Before stating these two versions of HBT, let us recall the notion of hyperplane in a vector space (we always consider vector spaces over the field $\mathbb{K}$ which is either $\mathbb{R}$ or $\mathbb{C}$). A *hyperplane* H in a vector space X over $\mathbb{K}$ is a maximal proper linear subspace of X or, equivalently, a linear subspace of codimension one, i.e. $dim X/H = 1$. Another equivalent formulation is that a hyperplane is a set of the form $\varphi^{-1}(\{0\})$ for some linear functional $\varphi : X \to \mathbb{K}$ not identically zero. The translation by a non-null vector of a hyperplane will be called *affine hyperplane*.

Theorem 5.1.1 (Analytic form of Hahn-Banach thm (for seminormed spaces)). *Let p be a seminorm on a vector space X over $\mathbb{K}$, M a linear subspace of X, and f a linear functional on M such that*

$$|f(x)| \leq p(x), \forall\, x \in M. \tag{5.1}$$

There exists a linear functional $\tilde{f}$ on X such that $\tilde{f}(x) = f(x), \forall\, x \in M$ and

$$|\tilde{f}(x)| \leq p(x), \forall\, x \in X. \tag{5.2}$$

Theorem 5.1.2 (Geometric form of Hahn-Banach theorem).
Let X be a topological vector space over $\mathbb{K}$, N a linear subspace of X, and Ω a convex open subset of X such that $N \cap \Omega = \emptyset$. Then there exists a closed hyperplane H of X such that

$$N \subseteq H \quad \text{and} \quad H \cap \Omega = \emptyset. \tag{5.3}$$

It should be remarked that the vector space X does not apparently carry any topology in Theorem 5.1.1, but actually the datum of a seminorm on X is equivalent to the datum of the topology induced by this seminorm. It is then clear that the conditions (5.1) and (5.2) imply the $p-$continuity of the functions f and $\tilde{f}$, respectively.

Let us also stress the fact that in Theorem 5.1.2 neither local convexity nor the Hausdorff separation property are assumed on the t.v.s. X. Moreover, it is easy to see that the geometric form of HBT could have been stated also in an affine setting, namely starting with any affine submanifold N of X which does not intersect the open convex subset Ω and getting a closed affine hyperplane fulfilling (5.3).

We will first show how to derive Theorem 5.1.1 from Theorem 5.1.2 and then give a proof of Theorem 5.1.2.

Before starting the proofs, let us fix one more definition. A *convex cone* C in a vector space X over $\mathbb{R}$ is a subset of X which is closed under addition and multiplication by positive scalars.

Proof. Theorem 5.1.2 $\Rightarrow$ Theorem 5.1.1
Let p be a seminorm on the vector space X, M a linear subspace of X, and f a linear functional defined on M fulfilling (5.1). As already remarked before, this means that f is continuous on M w.r.t. the topology induced by p on X (which makes X a l.c. t.v.s.).

Consider the subset $N := \{x \in M : f(x) = 1\}$. Taking any vector $x_0 \in N$, it is easy to see that $N - x_0 = Ker(f)$ (i.e. the kernel of f in M), which is a hyperplane of M and so a linear subspace of X. Therefore, setting $M_0 := N - x_0$, we have the following decomposition of M:

$$M = M_0 \oplus \mathbb{K}x_0,$$

where $\mathbb{K}x_0$ is the one-dimensional linear subspace through x_0. In other words

$$\forall\, x \in M,\ \exists!\, \lambda \in \mathbb{K}, y \in M_0 :\ x = y + \lambda x_0.$$

Then

$$\forall\, x \in M,\ f(x) = f(y) + \lambda f(x_0) = \lambda f(x_0) = \lambda,$$

which means that the values of f on M are completely determined by the ones on N. Consider now the open unit semiball of p:

$$U := \mathring{U}_p = \{x \in X : p(x) < 1\},$$

which we know being an open convex subset of X endowed with the topology induced by p. Then $N \cap U = \emptyset$ because if there was $x \in N \cap U$ then $p(x) < 1$ and $f(x) = 1$, which contradict (5.1).

By Theorem 5.1.2 (affine version), there exists a closed affine hyperplane H of X with the property that $N \subseteq H$ and $H \cap U = \emptyset$. Then $H - x_0$ is a hyperplane and so the kernel of a continuous linear functional $\tilde{f}$ on X non-identically zero.

Arguing as before (consider here the decomposition $X = (H - x_0) \oplus \mathbb{K}x_0$), we can deduce that the values of $\tilde{f}$ on X are completely determined by the ones on N and so on H (because for any $h \in H$ we have $h - x_0 \in Ker(\tilde{f})$ and so $\tilde{f}(h) - \tilde{f}(x_0) = \tilde{f}(h - x_0) = 0$). Since $\tilde{f} \not\equiv 0$, we have that $\tilde{f}(x_0) \neq 0$ and w.l.o.g. we can assume $\tilde{f}(x_0) = 1$ i.e. $\tilde{f} \equiv 1$ on H. Therefore, for any $x \in M$ there exist unique $\lambda \in \mathbb{K}$ and $y \in N - x_0 \subseteq H - x_0$ s.t. $x = y + \lambda x_0$, we get that:

$$\tilde{f}(x) = \lambda \tilde{f}(x_0) = \lambda = \lambda f(x_0) = f(x),$$

i.e. f is the restriction of $\tilde{f}$ to M. Furthermore, the fact that $H \cap U = \emptyset$ means that $\tilde{f}(x) = 1$ implies $p(x) \geq 1$. Then for any $y \in X$ s.t. $\tilde{f}(y) \neq 0$ we have that: $\tilde{f}\left(\frac{y}{\tilde{f}(y)}\right) = 1$ and so that $p\left(\frac{y}{\tilde{f}(y)}\right) \geq 1$ which implies that $|\tilde{f}(y)| \leq p(y)$. The latter obviously holds for $\tilde{f}(y) = 0$. Hence, (5.2) is established. □