which means that the values of f on M are completely determined by the ones on N. Consider now the open unit semiball of p:

$$U := \mathring{U}_p = \{x \in X : p(x) < 1\},$$

which we know being an open convex subset of X endowed with the topology induced by p. Then $N \cap U = \emptyset$ because if there was $x \in N \cap U$ then $p(x) < 1$ and $f(x) = 1$, which contradict (5.1).

By Theorem 5.1.2 (affine version), there exists a closed affine hyperplane H of X with the property that $N \subseteq H$ and $H \cap U = \emptyset$. Then $H - x_0$ is a hyperplane and so the kernel of a continuous linear functional $\tilde{f}$ on X non-identically zero.

Arguing as before (consider here the decomposition $X = (H - x_0) \oplus \mathbb{K}x_0$), we can deduce that the values of $\tilde{f}$ on X are completely determined by the ones on N and so on H (because for any $h \in H$ we have $h - x_0 \in Ker(\tilde{f})$ and so $\tilde{f}(h) - \tilde{f}(x_0) = \tilde{f}(h - x_0) = 0$). Since $\tilde{f} \not\equiv 0$, we have that $\tilde{f}(x_0) \neq 0$ and w.l.o.g. we can assume $\tilde{f}(x_0) = 1$ i.e. $\tilde{f} \equiv 1$ on H. Therefore, for any $x \in M$ there exist unique $\lambda \in \mathbb{K}$ and $y \in N - x_0 \subseteq H - x_0$ s.t. $x = y + \lambda x_0$, we get that:

$$\tilde{f}(x) = \lambda \tilde{f}(x_0) = \lambda = \lambda f(x_0) = f(x),$$

i.e. f is the restriction of $\tilde{f}$ to M. Furthermore, the fact that $H \cap U = \emptyset$ means that $\tilde{f}(x) = 1$ implies $p(x) \geq 1$. Then for any $y \in X$ s.t. $\tilde{f}(y) \neq 0$ we have that: $\tilde{f}\left(\frac{y}{\tilde{f}(y)}\right) = 1$ and so that $p\left(\frac{y}{\tilde{f}(y)}\right) \geq 1$ which implies that $|\tilde{f}(y)| \leq p(y)$. The latter obviously holds for $\tilde{f}(y) = 0$. Hence, (5.2) is established. □

Proof. Theorem 5.1.2
We assume that $\Omega \neq \emptyset$, otherwise there is nothing to prove.

1) Existence of a linear subspace H of X maximal for (5.3).
This first part of the proof is quite simple and consists in a straightforward application of Zorn's lemma. In fact, consider the family $\mathcal{F}$ of all the linear subspaces L of X such that

$$N \subseteq L \quad \text{and} \quad L \cap \Omega = \emptyset. \tag{5.4}$$

$\mathcal{F}$ is clearly non-empty since N belongs to it by assumption. If we take now a totally ordered subfamily $\mathcal{C}$ of $\mathcal{F}$ (totally ordered for the inclusion relation $\subseteq$), then the union of all the linear subspaces belonging to $\mathcal{C}$ is a linear subspace of X having the properties in (5.4). Hence, we can apply Zorn's lemma applies and conclude that there exists at least a maximal element H in $\mathcal{F}$.

2) H is closed in X.
The fact that H and Ω do not intersect gives that H is contained in the complement of Ω in X. This implies that also its closure $\overline{H}$ does not intersect Ω. Indeed, since Ω is open, we get

$$\overline{H} \subseteq \overline{X \setminus \Omega} = X \setminus \Omega.$$

Then $\overline{H}$ is a linear subspace (as closure of a linear subspace) of X, which is disjoint from Ω and which contains H and so N, i.e. $\overline{H} \in \mathcal{F}$. Hence, as H is maximal in $\mathcal{F}$, it must coincide with its closure. Note that the fact that H is closed guarantees that the quotient space X/H is a Hausdorff t.v.s. (see Proposition 2.3.5).

3) H is an hyperplane
We want to show that H is a hyperplane, i.e. that $dim(X/H) = 1$. To this aim we distinguish the two cases when $\mathbb{K} = \mathbb{R}$ and when $\mathbb{K} = \mathbb{C}$.

3.1) Case $\mathbb{K} = \mathbb{R}$
Let $\phi : X \to X/H$ be the canonical map. Since ϕ is an open linear mapping (see Proposition 2.3.2), $\phi(\Omega)$ is an open convex subset of X/H. Also we have that $\phi(\Omega)$ does not contain the origin $\hat{o}$ of X/H. Indeed, if $\hat{o} \in \phi(\Omega)$ holds, then there would exist $x \in \Omega$ s.t. $\phi(x) = \hat{o}$ and so $x \in H$, which would contradict the assumption $H \cap \Omega = \emptyset$. Let us set:

$$A = \bigcup_{\lambda > 0} \lambda\phi(\Omega).$$

Then the subset A of X/H is an open convex cone which does not contain the origin $\hat{o}$.

Let us observe that X/H has at least dimension 1. Indeed, if $dim(X/H) = 0$ then $X/H = \{\hat{o}\}$ and so $X = H$ which contradicts the fact that Ω does not intersect H (recall that we assumed Ω is non-empty). Suppose that $dim(X/H) \geq 2$, then to get our conclusion it will suffice to show the following claims:

Claim 1: The boundary ∂A of A must contain at least one point $x \neq \hat{o}$.

Claim 2: The point $-x$ cannot belong to A.

In fact, once Claim 1 is established, we have that $x \notin A$, because $x \in \partial A$ and A is open. This together with Claim 2 gives that both x and $-x$ belong to the complement of A in X/H and, therefore, so does the straight line L defined by these two points. (If there was a point $y \in L \cap A$ then any positive multiple of y would belong to $L \cap A$, as A is a cone. Hence, for some $\lambda > 0$ we would have $x = \lambda y \in L \cap A$, which contradicts the fact that $x \notin A$.) Then:

- $\phi^{-1}(L)$ is a linear subspace of X

- $\phi^{-1}(L) \cap \Omega = \emptyset$, since $L \cap A = \emptyset$
- $\phi^{-1}(L) \supsetneq H$ because $\hat{o} = \phi(H) \subseteq L$ but $L \neq \{\hat{o}\}$ since $x \neq \hat{o}$ is in L.

This contradicts the maximality of H and so $dim(X/H) = 1$.
To complete the proof of 3.1) let us show the two claims.

Proof. of Claim 1
Suppose that $\partial A = \{\hat{o}\}$. This means that A has empty boundary in the set $(X/H) \setminus \{\hat{o}\}$ and so that A is a connected component of $(X/H) \setminus \{\hat{o}\}$. Since $dim(X/H) \geq 2$, the space $(X/H) \setminus \{\hat{o}\}$ is arc-connected and so it is itself a connected space. Hence, $A = (X/H) \setminus \{\hat{o}\}$ which contradicts the convexity of A since $(X/H) \setminus \{\hat{o}\}$ is clearly not convex. □

Proof. of Claim 2
Suppose $-x \in A$. Then, as A is open, there is a neighborhood V of $-x$ entirely contained in A. This implies that $-V$ is a neighborhood of x. Since x is a boundary point of A, there exists $y \in (-V) \cap A$. But then $-y \in V \subset A$ and so, by the convexity of A, the whole line segment between y and $-y$ is contained in A, in particular $\hat{o}$, which contradicts the definition of A. □

3.2) Case $\mathbb{K} = \mathbb{C}$
Although here we are considering the scalars to be the complex numbers, we may view X as a vector space over the real numbers and it is obvious that its topology, as originally given, is still compatible with its linear structure. By step 3.1) above, we know that there exists a real hyperplane H_0 of X which contains N and does not intersect Ω. By a real hyperplane, we mean that H_0 is a linear subspace of X viewed as a vector space over the field of real numbers such that $dim_{\mathbb{R}}(X/H_0) = 1$.

Now it is easy to see that $iN = N$ (here $i = \sqrt{-1}$). Hence, setting $H := H_0 \cap iH_0$, we have that $N \subseteq H$ and $H \cap \Omega = \emptyset$. Then to complete the proof it remains to show that this H is a complex hyperplane. It is obviously a complex linear subspace of X and its real codimension is ≥ 1 and ≤ 2 (since the intersection of two distinct hyperplanes is always a linear subspace with codimension two). Hence, its complex codimension is equal to one.

□

5.2 Applications of Hahn-Banach theorem

The Hahn-Banach theorem is frequently applied in analysis, algebra and geometry, as will be seen in the forthcoming course. We will briefly indicate in this section some applications of this theorem to problems of separation of convex sets and to the multivariate moment problem. From now on we will focus on t.v.s. over the field of real numbers.

5.2.1 Separation of convex subsets of a real t.v.s.

Let X t.v.s. over the field of real numbers and H be a closed affine hyperplane of X. We say that two disjoint subsets A and B of X are *separated* by H if A is contained in one of the two closed half-spaces determined by H and B is contained in the other one. We can express this property in terms of functionals. Indeed, since $H = L^{-1}(\{a\})$ for some $L : X \to \mathbb{R}$ linear not identically zero and some $a \in \mathbb{R}$, we can write that A and B are separated by H if and only if:

$$\exists\, a \in \mathbb{R} \text{ s.t. } L(A) \geq a \text{ and } L(B) \leq a.$$

where for any $S \subseteq X$ the notation $L(S) \leq a$ simply means $\forall s \in S, L(s) \leq a$ (and analogously for $\geq, <, >, =, \neq$).
We say that A and B are *strictly separated* by H if at least one of the two inequalities is strict. (Note that there are several definition in literature for the strict separation but for us it will be just the one defined above) In the present subsection we would like to investigate whether one can separate, or strictly separate, two disjoint convex subsets of a real t.v.s..

Proposition 5.2.1. *Let X be a t.v.s. over the real numbers and A, B two disjoint convex subsets of X.*

a) If A is open nonempty and B is nonempty, then there exists a closed affine hyperplane H of X separating A and B, i.e. there exists $a \in \mathbb{R}$ and a functional $L : X \to \mathbb{R}$ linear not identically zero s.t. $L(A) \geq a$ and $L(B) \leq a$.

b) If in addition B is open, the hyperplane H can be chosen so as to strictly separate A and B, i.e. there exists $a \in \mathbb{R}$ and $L : X \to \mathbb{R}$ linear not identically zero s.t. $L(A) \geq a$ and $L(B) < a$.

c) If in addition A is a cone, then a can be chosen to be zero, i.e. there exists $L : X \to \mathbb{R}$ linear not identically zero s.t. $L(A) \geq 0$ and $L(B) < 0$.

Proof.

a) Consider the set $A - B := \{a - b : a \in A, b \in B\}$. Then: $A - B$ is an open subset of X as it is the union of the open sets $A - y$ as y varies over B; $A - B$ is convex as it is the Minkowski sum of the convex sets A and $-B$; and $o \notin (A - B)$ because if this was the case then there would be at least a point in the intersection of A and B which contradicts the assumption that they are disjoint. By applying Theorem 5.1.2 to $N = \{o\}$ and $U = A - B$ we have that there is a closed hyperplane H of X which does not intersect $A-B$ (and passes through the origin) or, which is equivalent, there exists a linear form f on X not identically zero such that $f(A-B) \neq 0$. Then there exists a linear form L on X not identically zero such that $L(A - B) > 0$ (in the case $f(A - B) < 0$ just take $L := -f$) i.e.

$$\forall\, x \in A, \forall\, y \in B, \;\; L(x) > L(y). \tag{5.5}$$

Since $B \neq \emptyset$ we have that $a := \inf_{x \in A} L(x) > -\infty$. Then (5.5) implies that $L(B) \leq a$ and we clearly have $L(A) \geq a$.

b) Let now both A and B be open convex and nonempty disjoint subsets of X. By part a) we have that there exists $a \in \mathbb{R}$ and $L : X \to \mathbb{R}$ linear not identically zero s.t. $L(A) \geq a$ and $L(B) \leq a$. Suppose that there exists $b \in B$ s.t. $L(b) = a$. Since B is open, for any $x \in X$ there exists $\varepsilon > 0$ s.t. for all $t \in [0, \varepsilon]$ we have $b + tx \in B$. Therefore, as $L(B) \leq a$, we have that

$$L(b + tx) \leq a, \forall\, t \in [0, \varepsilon]. \tag{5.6}$$

Now fix $x \in X$, consider the function $f(t) := L(b+tx)$ for all $t \in \mathbb{R}$ whose first derivative is clearly given by $f'(t) = L(x)$ for all $t \in \mathbb{R}$. Then (5.6) means that $t = 0$ is a point of local maximum for f and so $f'(0) = 0$ i.e. $L(x) = 0$. As x is an arbitrary point of x, we get $L \equiv 0$ on X which is a contradiction. Hence, $L(B) < a$.

c) Let now A be an open nonempty convex cone of X and B an open convex nonempty subset of X s.t. $A \cap B = \emptyset$. By part a) we have that there exists $a \in \mathbb{R}$ and $L : X \to \mathbb{R}$ linear not identically zero s.t. $L(A) \geq a$ and $L(B) \leq a$. Since A is a cone, for any $t > 0$ we have that $tA \subseteq A$ and so $tL(A) = L(tA) \geq a$ i.e. $L(A) \geq \frac{a}{t}$. This implies that $L(A) \geq \inf_{t>0} \frac{a}{t} = 0$. Moreover, part a) also gives that $L(B) < L(A)$. Therefore, for any $t > 0$ and any $x \in A$, we have in particular $L(B) < L(tx) = tL(x)$ and so $L(B) \leq \inf_{t>0} tL(x) = 0$. Since B is also open, we can exactly proceed as in part b) to get $L(B) < 0$.

□

Let us show now two interesting consequences of this result which we will use in the following subsection.

Corollary 5.2.2. *Let X be a vector space over $\mathbb{R}$ endowed with the finest locally convex topology φ. If C is a nonempty closed convex cone in X and $x_0 \in X \setminus C$ then there exists a linear functional $L : X \to \mathbb{R}$ non identically zero s.t. $L(C) \geq 0$ and $L(x_0) < 0$.*