The Hausdorff criterion could be paraphrased by saying that smaller neighborhoods make larger topologies. This is a very intuitive theorem, because the smaller the neighbourhoods are the easier it is for a set to contain neighbourhoods of all its points and so the more open sets there will be.

Proof.

$\Rightarrow$ Suppose $\tau \subseteq \tau'$. Fixed any point $x \in X$, let $U \in \mathcal{B}(x)$. Then, since U is a neighbourhood of x in (X, τ), there exists $O \in \tau$ s.t. $x \in O \subseteq U$. But $O \in \tau$ implies by our assumption that $O \in \tau'$, so U is also a neighbourhood of x in (X, τ'). Hence, by Def. 1.1.11 for $\mathcal{B}'(x)$, there exists $V \in \mathcal{B}'(x)$ s.t. $V \subseteq U$.

$\Leftarrow$ Conversely, let $W \in \tau$. Then W is a neighbourhood of x w.r.t. τ. Since $\mathcal{B}(x)$ is a base of neighbourhoods w.r.t. τ, for each $x \in W$ there exists $U \in \mathcal{B}(x)$ such that $x \in U \subseteq W$. This together with the assumption guarantees that there exists $V \in \mathcal{B}'(x)$ s.t. $x \in V \subseteq U \subseteq W$. Hence, by Remark 1.1.10, we have $W \in \tau'$. $\square$

1.1.3 Reminder of some simple topological concepts

Definition 1.1.18. *Given a topological space (X, τ) and a subset S of X, the* subset or induced topology *on S is defined by $\tau_S := \{S \cap U \mid U \in \tau\}$. That is, a subset of S is open in the subset topology if and only if it is the intersection of S with an open set in (X, τ).*
Alternatively, we can define the subspace topology for a subset S of X as the coarsest topology for which the inclusion map $\iota : S \hookrightarrow X$ is continuous.

Note that (S, τ_s) is a topological space in its own.

Definition 1.1.19. *Given a collection of topological space (X_i, τ_i), where $i \in I$ (I is an index set possibly uncountable), the* product topology *on the Cartesian product $X := \prod_{i \in I} X_i$ is defined in the following way: a set U is open in X iff it is an arbitrary union of sets of the form $\prod_{i \in I} U_i$, where each $U_i \in \tau_i$ and $U_i \neq X_i$ for only finitely many i.*
Alternatively, we can define the product topology to be the coarsest topology for which all the canonical projections $p_i : X \to X_i$ are continuous.

Given a topological space X, we define:

Definition 1.1.20.

- *The* closure *of a subset $A \subseteq X$ is the smallest closed set containing A. It will be denoted by $\bar{A}$. Equivalently, $\bar{A}$ is the intersection of all closed subsets of X containing A.*
- *The* interior *of a subset $A \subseteq X$ is the largest open set contained in it. It will be denoted by $\mathring{A}$. Equivalently, $\mathring{A}$ is the union of all open subsets of X contained in A.*

Proposition 1.1.21. *Given a top. space X and $A \subseteq X$, the following hold.*

- *A point x is a* closure point *of A, i.e. $x \in \bar{A}$, if and only if each neighborhood of x has a nonempty intersection with A.*
- *A point x is an* interior point *of A, i.e. $x \in \mathring{A}$, if and only if there exists a neighborhood of x which entirely lies in A.*
- *A is closed in X iff $A = \bar{A}$.*
- *A is open in X iff $A = \mathring{A}$.*

Proof. (Sheet 2, Exercise 1)

Example 1.1.22. *Let τ be the standard euclidean topology on $\mathbb{R}$. Consider $X := (\mathbb{R}, \tau)$ and $Y := \big((0,1], \tau_Y\big)$, where τ_Y is the topology induced by τ on $(0,1]$. The closure of $(0, \frac{1}{2})$ in X is $[0, \frac{1}{2}]$, but its closure in Y is $(0, \frac{1}{2}]$.*

Definition 1.1.23. *Let A and B be two subsets of the same topological space X. A is* dense *in B if $B \subseteq \bar{A}$. In particular, A is said to be dense in X (or* everywhere dense*) if $\bar{A} = X$.*

Examples 1.1.24.

- *Standard examples of sets everywhere dense in the real line $\mathbb{R}$ (with the euclidean topology) are the set of rational numbers $\mathbb{Q}$ and the one of irrational numbers $\mathbb{R} - \mathbb{Q}$.*
- *A set X is equipped with the discrete topology if and only if the whole space X is the only dense set in itself.*
  If X has the discrete topology then every subset is equal to its own closure (because every subset is closed), so the closure of a proper subset is always proper. Conversely, if X is the only dense subset of itself, then for every proper subset A its closure $\bar{A}$ is also a proper subset of X. Let $y \in X$ be arbitrary. Then to $\overline{X \setminus \{y\}}$ is a proper subset of X and so it has to be equal to its own closure. Hence, $\{y\}$ is open. Since y is arbitrary, this means that X has the discrete topology.
- *Every non-empty subset of a set X equipped with the trivial topology is dense, and every topology for which every non-empty subset is dense must be trivial.*
  If X has the trivial topology and A is any non-empty subset of X, then the only closed subset of X containing A is X. Hence, $\bar{A} = X$, i.e. A is dense in X. Conversely, if X is endowed with a topology τ for which every non-empty subset is dense, then the only non-empty subset of X which is closed is X itself. Hence, $\emptyset$ and X are the only closed subsets of τ. This means that X has the trivial topology.

Proposition 1.1.25. *Let X be a topological space and $A \subset X$. A is dense in X if and only if every nonempty open set in X contains a point of A.*

Proof. If A is dense in X, then by definition $\bar{A} = X$. Let O be any nonempty open subset in X. Then for any $x \in O$ we have that $x \in \bar{A}$ and $O \in \mathcal{F}(x)$. Therefore, by Proposition 1.1.21, we have that $O \cap A \neq \emptyset$. Conversely, let $x \in X$. By definition of neighbourhood, for any $U \in \mathcal{F}(x)$ there exists an open subset O of X s.t. $x \in O \subseteq U$. Then $U \cap A \neq \emptyset$ since O contains a point of A by our assumption. Hence, by Proposition 1.1.21, we get $x \in \bar{A}$ and so that A is dense in X. □

Definition 1.1.26. *A topological space X is said to be separable if there exists a countable dense subset of X.*

Example 1.1.27.

- $\mathbb{R}$ *with the euclidean topology is separable.*
- *The space $\mathcal{C}([0,1])$ of all continuous functions from $[0,1]$ to $\mathbb{R}$ endowed with the uniform topology*[1] *is separable, since by the Weirstrass approximation theorem $\overline{\mathbb{Q}[x]} = \mathcal{C}([0,1])$.*

Let us briefly consider now the notion of convergence.
First of all let us concern with filters. When do we say that a filter $\mathcal{F}$ on a topological space X converges to a point $x \in X$? Intuitively, if $\mathcal{F}$ has to converge to x, then the elements of $\mathcal{F}$, which are subsets of X, have to get somehow "smaller and smaller" about x, and the points of these subsets need to get "nearer and nearer" to x. This can be made more precise by using neighborhoods of x: we want to formally express the fact that, however small a neighborhood of x is, it should contain some subset of X belonging to the filter $\mathcal{F}$ and, consequently, all the elements of $\mathcal{F}$ which are contained in that particular one. But in view of Axiom (F3), this means that the neighborhood of x under consideration must itself belong to the filter $\mathcal{F}$, since it must contain some element of $\mathcal{F}$.

Definition 1.1.28. *Given a filter $\mathcal{F}$ in a topological space X, we say that it converges to a point $x \in X$ if every neighborhood of x belongs to $\mathcal{F}$, in other words if $\mathcal{F}$ is finer than the filter of neighborhoods of x.*

We recall now the definition of convergence of a sequence to a point and we see how it easily connects to the previous definition.

[1]The uniform topology on $\mathcal{C}([0,1])$ is the topology induced by the supremum norm $\|\cdot\|_\infty$, i.e. the topology on $\mathcal{C}([0,1])$ having as basis of neighbourhoods of any $f \in \mathcal{C}([0,1])$ the collection $\{B_\varepsilon(f) : \varepsilon \in \mathbb{R}^+\}$ where $B_\varepsilon(f) := \{g \in \mathcal{C}([0,1]) : \|g - f\|_\infty < \varepsilon\}$ and $\|h\|_\infty := \sup_{x\in[0,1]} |h(x)|, \forall h \in \mathcal{C}([0,1])$

Definition 1.1.29. *Given a sequence of points $\{x_n\}_{n\in\mathbb{N}}$ in a topological space X, we say that it converges to a point $x \in X$ if for any $U \in \mathcal{F}(x)$ there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.*

If we now consider the filter $\mathcal{F}_S$ associated to the sequence $S := \{x_n\}_{n\in\mathbb{N}}$, i.e. $\mathcal{F}_S := \{A \subset X : |S \setminus A| < \infty\}$, then it is easy to see that:

Proposition 1.1.30. *Given a sequence of points $S := \{x_n\}_{n\in\mathbb{N}}$ in a topological space X, S converges to a point $x \in X$ if and only if the associated filter $\mathcal{F}_S$ converges to x.*

Proof. Set for each $m \in \mathbb{N}$, set $S_m := \{x_n \in S : n \geq m\}$. By Definition 1.1.29, S converges to x iff $\forall\, U \in \mathcal{F}(x), \exists N \in \mathbb{N} : S_N \subseteq U$. As $\mathcal{B} := \{S_m : m \in \mathbb{N}\}$ is a basis for $\mathcal{F}_S$ (see Problem Sheet 1, Exercise 2 c)), we have that $\forall\, U \in \mathcal{F}(x), U \in \mathcal{F}_S$, which is equivalent to say that $\mathcal{F}(x) \subseteq \mathcal{F}_S$. □

1.1.4 Mappings between topological spaces

Let (X, τ_X) and (Y, τ_Y) be two topological spaces.

Definition 1.1.31. *A map $f : X \to Y$ is* continuous *if the preimage of any open set in Y is open in X, i.e. $\forall\, U \in \tau_Y$, $f^{-1}(U) := \{x \in X : f(x) \in U\} \in \tau_X$. Equivalently, given any point $x \in X$ and any $V \in \mathcal{F}(f(x))$ in Y, the preimage $f^{-1}(V) \in \mathcal{F}(x)$ in X.*

Examples 1.1.32.

- *Any constant map $f : X \to Y$ is continuous.*
  Suppose that $f(x) := y$ for all $x \in X$ and some $y \in Y$. Let $U \in \tau_Y$. If $y \in U$ then $f^{-1}(U) = X$ and if $y \notin U$ then $f^{-1}(U) = \emptyset$. Hence, in either case, $f^{-1}(U)$ is open in τ_X.
- *If $g : X \to Y$ is continuous, then the restriction of g to any subset S of X is also continuous w.r.t. the subset topology induced on S by the topology on X.*
- *Let X be a set endowed with the discrete topology, Y be a set endowed with the trivial topology and Z be any topological space. Any maps $f : X \to Z$ and $g : Z \to Y$ are continuous.*

Definition 1.1.33. *A mapping $f : X \to Y$ is* open *if the image of any open set in X is open in Y, i.e. $\forall\, V \in \tau_X$, $f(V) := \{f(x) : x \in V\} \in \tau_Y$. In the same way, a* closed mapping *$f : X \to Y$ sends closed sets to closed sets.*

Note that a map may be open, closed, both, or neither of them. Moreover, open and closed maps are not necessarily continuous.

Example 1.1.34. *If Y has the discrete topology (i.e. all subsets are open and closed) then every function $f : X \to Y$ is both open and closed (but not necessarily continuous). For example, if we take the standard euclidean topology on $\mathbb{R}$ and the discrete topology on $\mathbb{Z}$ then the floor function $\mathbb{R} \to \mathbb{Z}$ is open and closed, but not continuous. (Indeed, the preimage of the open set $\{0\}$ is $[0, 1) \subset \mathbb{R}$, which is not open in the standard euclidean topology).*

If a continuous map f is one-to-one, f^{-1} does not need to be continuous.

Example 1.1.35.
Let us consider $[0, 1) \subset \mathbb{R}$ and $S^1 \subset \mathbb{R}^2$ endowed with the subspace topologies given by the euclidean topology on $\mathbb{R}$ and on $\mathbb{R}^2$, respectively. The map

$$\begin{array}{rcc} f : [0,1) & \to & S^1 \\ t & \mapsto & (\cos 2\pi t, \sin 2\pi t). \end{array}$$

is bijective and continuous but f^{-1} is not continuous, since there are open subsets of $[0, 1)$ whose image under f is not open in S^1. (For example, $[0, \frac{1}{2})$ is open in $[0, 1)$ but $f([0, \frac{1}{2}))$ is not open in S^1.)

Definition 1.1.36. *A one-to-one map f from X onto Y is a* homeomorphism *if and only if f and f^{-1} are both continuous. Equivalently, iff f and f^{-1} are both open (closed). If such a mapping exists, X and Y are said to be two* homeomorphic topological spaces*.*

In other words an homeomorphism is a one-to-one mapping which sends every open (resp. closed) set of X in an open (resp. closed) set of Y and viceversa, i.e. an homeomorphism is both an open and closed map. Note that the homeomorphism gives an equivalence relation on the class of all topological spaces.

Examples 1.1.37. *In these examples we consider any subset of $\mathbb{R}^n$ endowed with the subset topology induced by the Euclidean topology on $\mathbb{R}^n$.*

1. *Any open interval of $\mathbb{R}$ is homeomorphic to any other open interval of $\mathbb{R}$ and also to $\mathbb{R}$ itself.*
2. *A circle and a square in $\mathbb{R}^2$ are homeomorphic.*
3. *The circle S^1 with a point removed is homeomorphic to $\mathbb{R}$.*

Let us consider now the case when a set X carries two different topologies τ_1 and τ_2. Then the following two properties are equivalent:

- the identity ι of X is continuous as a mapping from (X, τ_1) and (X, τ_2)
- the topology τ_1 is finer than the topology τ_2.

Therefore, ι is a homeomorphism if and only if the two topologies coincide.

Proof. Suppose that ι is continuous. Let $U \in \tau_2$. Then $\iota^{-1}(U) = U \in \tau_1$, hence $U \in \tau_1$. Therefore, $\tau_2 \subseteq \tau_1$. Conversely, assume that $\tau_2 \subseteq \tau_1$ and take any $U \in \tau_2$. Then $U \in \tau_1$ and by definition of identity we know that $\iota^{-1}(U) = U$. Hence, $\iota^{-1}(U) \in \tau_1$ and therefore, ι is continuous. □

Proposition 1.1.38. *Continuous maps preserve the convergence of sequences. That is, if $f : X \to Y$ is a continuous map between two topological spaces (X, τ_X) and (Y, τ_Y) and if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of points in X convergent to a point $x \in X$ then $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $f(x) \in Y$.*

Proof. (Sheet 2, Exercise 4 b)) □

1.1.5 Hausdorff spaces

Definition 1.1.39. *A topological space X is said to be* Hausdorff *(or* separated*) if any two distinct points of X have neighbourhoods without common points; or equivalently if:*
(T2) two distinct points always lie in disjoint open sets.

In literature, the Hausdorff space are often called *T2-spaces* and the axiom (T2) is said to be the *separation axiom*.

Proposition 1.1.40. *In a Hausdorff space the intersection of all closed neighbourhoods of a point contains the point alone. Hence, the singletons are closed.*

Proof. Let us fix a point $x \in X$, where X is a Hausdorff space. Denote by C the intersection of all closed neighbourhoods of x. Suppose that there exists $y \in C$ with $y \neq x$. By definition of Hausdorff space, there exist a neighbourhood $U(x)$ of x and a neighbourhood $V(y)$ of y s.t. $U(x) \cap V(y) = \emptyset$. Therefore, $y \notin \overline{U(x)}$ because otherwise any neighbourhood of y (in particular $V(y)$) should have non-empty intersection with $U(x)$. Hence, $y \notin C$. □

Examples 1.1.41.

1. *Any metric space[2] is Hausdorff.*
   Indeed, for any $x, y \in (X, d)$ with $x \neq y$ just choose $0 < \varepsilon < \frac{1}{2}d(x, y)$ and you get $B_\varepsilon(x) \cap B_\varepsilon(y) = \emptyset$.
2. *Any set endowed with the discrete topology is a Hausdorff space.*
   Indeed, any singleton is open in the discrete topology so for any two distinct point x, y we have that $\{x\}$ and $\{y\}$ are disjoint and open.

[2] Any metric space (X, d) is a topological space, because we can equip it with the topology induced by the metric d, i.e. the topology having as basis of neighbourhoods of any $x \in X$ the collection $\{B_\varepsilon(x) : \varepsilon \in \mathbb{R}^+\}$ where $B_\varepsilon(x) := \{y \in X : d(y, x) < \varepsilon\}$.

3. *The only Hausdorff topology on a finite set is the discrete topology.*
   Let X be a finite set endowed with a Hausdorff topology τ. As X is finite, any subset S of X is finite and so S is a finite union of singletons. But since (X, τ) is Hausdorff, the previous proposition implies that any singleton is closed. Hence, any subset S of X is closed and so the τ has to be the discrete topology.
4. *An infinite set with the cofinite topology is not Hausdorff.*
   In fact, any two non-empty open subsets O_1, O_2 in the cofinite topology on X are complements of finite subsets. Therefore, their intersection $O_1 \cap O_2$ is a complement of a finite subset, but X is infinite and so $O_1 \cap O_2 \neq \emptyset$. Hence, X is not Hausdorff.

1.2 Linear mappings between vector spaces

The basic notions from linear algebra are assumed to be well-known and so they are not recalled here. However, we briefly give again the definition of vector space and fix some general terminology for linear mappings between vector spaces. In this section we are going to consider vector spaces over the field $\mathbb{K}$ of real or complex numbers which is given the usual euclidean topology defined by means of the modulus.

Definition 1.2.1. *A set X with the two mappings:*

$$\begin{array}{rcll} X \times X & \to & X & \\ (x, y) & \mapsto & x + y & \textit{vector addition} \end{array}$$

$$\begin{array}{rcll} \mathbb{K} \times X & \to & X & \\ (\lambda, x) & \mapsto & \lambda x & \textit{scalar multiplication} \end{array}$$

is a vector space *(or* linear space*) over $\mathbb{K}$ if the following axioms are satisfied:*

(L1)
1. $(x + y) + z = x + (y + z)$, $\forall\, x, y, z \in X$ *(associativity of +)*
2. $x + y = y + x$, $\forall\, x, y \in X$ *(commutativity of +)*
3. $\exists\, o \in X$: $x + o = x$, $\forall\, x, \in X$ *(neutral element for +)*
4. $\forall\, x \in X$, $\exists! -x \in X$ *s.t.* $x + (-x) = o$ *(inverse element for +)*

(L2)
1. $\lambda(\mu x) = (\lambda\mu)x$, $\forall\, x \in X$, $\forall\, \lambda, \mu \in \mathbb{K}$
   (compatibility of scalar multiplication with field multiplication)
2. $1x = x \,\forall\, x \in X$ *(neutral element for scalar multiplication)*
3. $(\lambda + \mu)x = \lambda x + \mu x$, $\forall\, x \in X$, $\forall\, \lambda, \mu \in \mathbb{K}$
   (distributivity of scalar multiplication with respect to field addition)
4. $\lambda(x + y) = \lambda x + \lambda y$, $\forall\, x, y \in X$, $\forall\, \lambda \in \mathbb{K}$
   (distributivity of scalar multiplication wrt vector addition)

Definition 1.2.2.
Let X, Y be two vector space over $\mathbb{K}$. A mapping $f : X \to Y$ is called linear mapping *or* homomorphism *if f preserves the vector space structure, i.e.* $f(\lambda x + \mu y) = \lambda f(x) + \mu f(y) \, \forall \, x, y \in X, \, \forall \, \lambda, \mu \in \mathbb{K}$.

Definition 1.2.3.

- *A linear mapping from X to itself is called* endomorphism.
- *A one-to-one linear mapping is called* monomorphism. *If S is a subspace of X, the identity map is a monomorphism and it is called* embedding.
- *An onto (surjective) linear mapping is called* epimorphism.
- *A bijective (one-to-one and onto) linear mapping between two vector spaces X and Y over $\mathbb{K}$ is called* (algebraic) isomorphism. *If such a map exists, we say that X and Y are (algebraically) isomorphic $X \cong Y$.*
- *An isomorphism from X into itself is called* automorphism.

It is easy to prove that: A linear mapping is one-to-one (injective) if and only if $f(x) = 0$ implies $x = 0$.

Definition 1.2.4. *A linear mapping from $X \to \mathbb{K}$ is called* linear functional *or* linear form *on X. The set of all linear functionals on X is called* algebraic dual *and it is denoted by X^*.*

Note that the dual space of a finite dimensional vector space X is isomorphic to X.