Chapter 2

Topological Vector Spaces

2.1 Definition and properties of a topological vector space

In this section we are going to consider vector spaces over the field $\mathbb{K}$ of real or complex numbers which is given the usual euclidean topology defined by means of the modulus.

Definition 2.1.1. *A vector space X over $\mathbb{K}$ is called a* topological vector space *(t.v.s.) if X is provided with a topology τ which is compatible with the vector space structure of X, i.e. τ makes the vector space operations both continuous.*

More precisely, the condition in the definition of t.v.s. requires that:

$$\begin{array}{ccccc} X \times X & \to & X & & \\ (x,y) & \mapsto & x+y & & \textit{vector addition} \end{array}$$

$$\begin{array}{ccccc} \mathbb{K} \times X & \to & X & & \\ (\lambda,x) & \mapsto & \lambda x & & \textit{scalar multiplication} \end{array}$$

are both continuous when we endow X with the topology τ, $\mathbb{K}$ with the euclidean topology, $X \times X$ and $\mathbb{K} \times X$ with the correspondent product topologies.

Remark 2.1.2. *If (X,τ) is a t.v.s then it is clear from Definition 2.1.1 that $\sum_{k=1}^{N} \lambda_k^{(n)} x_k^{(n)} \to \sum_{k=1}^{N} \lambda_k x_k$ as $n \to \infty$ w.r.t. τ if for each $k = 1,\dots,N$ as $n \to \infty$ we have that $\lambda_k^{(n)} \to \lambda_k$ w.r.t. the euclidean topology on $\mathbb{K}$ and $x_k^{(n)} \to x_k$ w.r.t. τ.*

Let us discuss now some examples and counterexamples of t.v.s.

Examples 2.1.3.

a) Every vector space X over $\mathbb{K}$ endowed with the trivial topology is a t.v.s..

b) Every normed vector space endowed with the topology given by the metric induced by the norm is a t.v.s. (Sheet 3, Exercise 1 a)).

c) There are also examples of spaces whose topology cannot be induced by a norm or a metric but that are t.v.s., e.g. the space of infinitely differentiable functions, the spaces of test functions and the spaces of distributions (we will see later in details their topologies).

In general, a metric vector space is not a t.v.s.. Indeed, there exist metrics for which both the vector space operations of sum and product by scalars are discontinuous (see Sheet 3, Exercise 1 c) for an example).

Proposition 2.1.4. *Every vector space X over $\mathbb{K}$ endowed with the discrete topology is not a t.v.s. unless $X = \{o\}$.*

Proof. Assume that it is a t.v.s. and take $o \neq x \in X$. The sequence $\alpha_n = \frac{1}{n}$ in $\mathbb{K}$ converges to 0 in the euclidean topology. Therefore, since the scalar multiplication is continuous, $\alpha_n x \to o$ by Proposition 1.1.38, i.e. for any neighbourhood U of o in X there exists $m \in \mathbb{N}$ s.t. $\alpha_n x \in U$ for all $n \geq m$. In particular, we can take $U = \{o\}$ since it is itself open in the discrete topology. Hence, $\alpha_m x = o$, which implies that $x = o$ and so a contradiction. □

Definition 2.1.5. *Two t.v.s. X and Y over $\mathbb{K}$ are* (topologically) isomorphic *if there exists a vector space isomorphism $X \to Y$ which is at the same time a homeomorphism (i.e. bijective, linear, continuous and inverse continuous).*

In analogy to Definition 1.2.3, let us collect here the corresponding terminology for mappings between two t.v.s..

Definition 2.1.6. *Let X and Y be two t.v.s. on $\mathbb{K}$.*

- *A* topological homomorphism *from X to Y is a linear mapping which is also continuous and open.*
- *A* topological monomorphism *from X to Y is an injective topological homomorphism.*
- *A* topological isomorphism *from X to Y is a bijective topological homomorphism.*
- *A* topological automorphism *of X is a topological isomorphism from X into itself.*

Proposition 2.1.7. *Given a t.v.s. X, we have that:*

1. *For any $x_0 \in X$, the mapping $x \mapsto x + x_0$ (*translation *by x_0) is a homeomorphism of X onto itself.*
2. *For any $0 \neq \lambda \in \mathbb{K}$, the mapping $x \mapsto \lambda x$ (*dilation *by λ) is a topological automorphism of X.*

Proof. Both mappings are continuous by the very definition of t.v.s.. Moreover, they are bijections by the vector space axioms and their inverses $x \mapsto x - x_0$ and $x \mapsto \frac{1}{\lambda}x$ are also continuous. Note that the second map is also linear so it is a topological automorphism. $\square$

Proposition 2.1.7–1 shows that the topology of a t.v.s. is always a *translation invariant topology*, i.e. all translations are homeomorphisms. Note that the translation invariance of a topology τ on a vector space X is not sufficient to conclude (X, τ) is a t.v.s..

Example 2.1.8. *If a metric d on a vector space X is translation invariant, i.e. $d(x+z, y+z) = d(x, y)$ for all $x, y \in X$ (e.g. the metric induced by a norm), then the topology induced by the metric is translation invariant and the addition is always continuous. However, the multiplication by scalars does not need to be necessarily continuous (take d to be the discrete metric, then the topology generated by the metric is the discrete topology which is not compatible with the scalar multiplication see Proposition 2.1.4).*

The translation invariance of the topology of a t.v.s. means, roughly speaking, that a t.v.s. X topologically looks about any point as it does about any other point. More precisely:

Corollary 2.1.9. *The filter $\mathcal{F}(x)$ of neighbourhoods of x in a t.v.s. X coincides with the family of the sets $O+x$ for all $O \in \mathcal{F}(o)$, where $\mathcal{F}(o)$ is the filter of neighbourhoods of the origin o (i.e. neutral element of the vector addition).*

Proof. (Sheet 3, Exercise 2 a)) $\square$

Thus the topology of a t.v.s. is completely determined by the filter of neighbourhoods of any of its points, in particular by the filter of neighbourhoods of the origin o or, more frequently, by a base of neighbourhoods of the origin o. Therefore, we need some criteria on a filter of a vector space X which ensures that it is the filter of neighbourhoods of the origin w.r.t. some topology compatible with the vector structure of X.

Theorem 2.1.10. *A filter $\mathcal{F}$ of a vector space X over $\mathbb{K}$ is the filter of neighbourhoods of the origin w.r.t. some topology compatible with the vector structure of X if and only if*

1. *The origin belongs to every set $U \in \mathcal{F}$*
2. *$\forall U \in \mathcal{F}$, $\exists V \in \mathcal{F}$ s.t. $V + V \subset U$*
3. *$\forall U \in \mathcal{F}$, $\forall \lambda \in \mathbb{K}$ with $\lambda \neq 0$ we have $\lambda U \in \mathcal{F}$*
4. *$\forall U \in \mathcal{F}$, U is absorbing.*
5. *$\forall U \in \mathcal{F}$, $\exists V \in \mathcal{F}$ balanced s.t. $V \subset U$.*

Before proving the theorem, let us fix some definitions and notations:

Definition 2.1.11. *Let U be a subset of a vector space X.*

1. *U is* absorbing *(or* radial*) if $\forall x \in X$ $\exists \rho > 0$ s.t. $\forall \lambda \in \mathbb{K}$ with $|\lambda| \leq \rho$ we have $\lambda x \in U$. Roughly speaking, we may say that a subset is absorbing if it can be made by dilation to swallow every point of the whole space.*
2. *U is* balanced *(or* circled*) if $\forall x \in U$, $\forall \lambda \in \mathbb{K}$ with $|\lambda| \leq 1$ we have $\lambda x \in U$. Note that the line segment joining any point x of a balanced set U to $-x$ lies in U.*

Clearly, o must belong to every absorbing or balanced set. The underlying field can make a substantial difference. For example, if we consider the closed interval $[-1, 1] \subset \mathbb{R}$ then this is a balanced subset of $\mathbb{C}$ as real vector space, but if we take $\mathbb{C}$ as complex vector space then it is not balanced. Indeed, if we take $i \in \mathbb{C}$ we get that $i1 = i \notin [-1, 1]$.

Examples 2.1.12.

a) *In a normed space the unit balls centered at the origin are absorbing and balanced.*

b) *The unit ball B centered at $(\frac{1}{2}, 0) \in \mathbb{R}^2$ is absorbing but not balanced in the real vector space $\mathbb{R}^2$.* Indeed, B is a neighbourhood of the origin and so by Theorem 2.1.10-4 is absorbing. However, B is not balanced because for example if we take $x = (1, 0) \in B$ and $\lambda = -1$ then $\lambda x \notin B$.

c) *In the real vector space $\mathbb{R}^2$ endowed with the euclidean topology, the subset in Figure 2.1 is absorbing and the one in Figure 2.2 is balanced.*

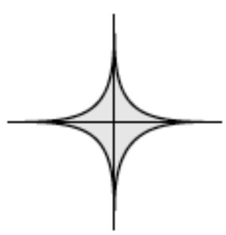

Figure 2.1: Absorbing

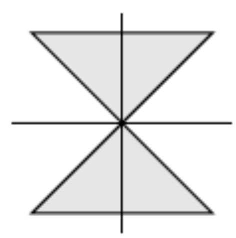

Figure 2.2: Balanced

d) *The polynomials $\mathbb{R}[x]$ are a balanced but not absorbing subset of the real space $\mathcal{C}([0, 1], \mathbb{R})$ of continuous real valued functions on $[0, 1]$.* Indeed, any multiple of a polynomial is still a polynomial but not every continuous function can be written as multiple of a polynomial.

e) *The subset* $A := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq |z_2|\}$ *of the complex space* $\mathbb{C}^2$ *endowed with the euclidean topology is balanced but* $\mathring{A}$ *is not balanced.* Indeed, $\forall\,(z_1, z_2) \in A$ and $\forall \lambda \in \mathbb{C}$ with $|\lambda| \leq 1$ we have that

$$|\lambda z_1| = |\lambda||z_1| \leq |\lambda||z_2| = |\lambda z_2|$$

i.e. $\lambda(z_1, z_2) \in A$. Hence, A is balanced. If we consider instead $\mathring{A} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2|\}$ then $\forall\,(z_1, z_2) \in \mathring{A}$ and $\lambda = 0$ we have that $\lambda(z_1, z_2) = (0, 0) \notin \mathring{A}$. Hence, $\mathring{A}$ is not balanced.

Proposition 2.1.13.
a) If B *is a balanced subset of a t.v.s.* X *then so is* $\bar{B}$*.*
b) If B *is a balanced subset of a t.v.s.* X *and* $o \in \mathring{B}$ *then* $\mathring{B}$ *is balanced.*

Proof. (Sheet 3, Exercise 2 b) c)) □

Proof. of Theorem 2.1.10.
Necessity part.
Suppose that X is a t.v.s. then we aim to show that the filter of neighbourhoods of the origin $\mathcal{F}$ satisfies the properties 1,2,3,4,5. Let $U \in \mathcal{F}$.

1. obvious, since every set $U \in \mathcal{F}$ is a neighbourhood of the origin o.
2. Since by the definition of t.v.s. the addition $(x, y) \mapsto x+y$ is a continuous mapping, the preimage of U under this map must be a neighbourhood of $(o, o) \in X \times X$. Therefore, it must contain a rectangular neighbourhood $W \times W'$ where $W, W' \in \mathcal{F}$. Taking $V = W \cap W'$ we get the conclusion, i.e. $V + V \subset U$.
3. By Proposition 2.1.7, fixed an arbitrary $0 \neq \lambda \in \mathbb{K}$, the map $x \mapsto \lambda^{-1}x$ of X into itself is continuous. Therefore, the preimage of any neighbourhood U of the origin must be also such a neighbourhood. This preimage is clearly λU, hence $\lambda U \in \mathcal{F}$.