2.3 Quotient topological vector spaces

Quotient topology

Let X be a topological space and $\sim$ be any equivalence relation on X. Then the *quotient set* $X/\sim$ is defined to be the set of all equivalence classes w.r.t. to $\sim$. The map $\phi : X \to X/\sim$ which assigns to each $x \in X$ its equivalence class $\phi(x)$ w.r.t. $\sim$ is called the *canonical map* or *quotient map*. Note that ϕ is surjective. We may define a topology on $X/\sim$ by setting that: a subset U of $X/\sim$ is open iff the preimage $\phi^{-1}(U)$ is open in X. This is called the *quotient topology* on $X/\sim$. Then it is easy to verify (Sheet 5, Exercise 2) that:

- the quotient map ϕ is continuous.
- the quotient topology on $X/\sim$ is the finest topology on $X/\sim$ such that ϕ is continuous.

Note that the quotient map ϕ is not necessarily open or closed.

Example 2.3.1. *Consider $\mathbb{R}$ with the standard topology given by the modulus and define the following equivalence relation on $\mathbb{R}$:*

$$x \sim y \Leftrightarrow (x = y \vee \{x, y\} \subset \mathbb{Z}).$$

Let $\mathbb{R}/\sim$ be the quotient set w.r.t $\sim$ and $\phi : \mathbb{R} \to \mathbb{R}/\sim$ the correspondent quotient map. Let us consider the quotient topology on $\mathbb{R}/\sim$. Then ϕ is not an open map. In fact, if U is an open proper subset of $\mathbb{R}$ containing an integer, then $\phi^{-1}(\phi(U)) = U \cup \mathbb{Z}$ which is not open in $\mathbb{R}$ with the standard topology. Hence, $\phi(U)$ is not open in $\mathbb{R}/\sim$ with the quotient topology.

For an example of quotient map which is not closed see Example 2.3.3 in the following.

Quotient vector space

Let X be a vector space and M a linear subspace of X. For two arbitrary elements $x, y \in X$, we define $x \sim_M y$ iff $x - y \in M$. It is easy to see that $\sim_M$ is an equivalence relation: it is reflexive, since $x - x = 0 \in M$ (every linear subspace contains the origin); it is symmetric, since $x - y \in M$ implies $-(x - y) = y - x \in M$ (a linear subspace contains all scalar multiples of every of its elements); it is transitive, since $x - y \in M$, $y - z \in M$ implies $x - z = (x - y) + (y - z) \in M$ (when a linear subspace contains two vectors, it also contains their sum). Then X/M is defined to be the quotient set $X/\sim_M$, i.e. the set of all equivalence classes for the relation $\sim_M$ described above. The canonical (or quotient) map $\phi : X \to X/M$ which assigns to each $x \in X$ its equivalence class $\phi(x)$ w.r.t. the relation $\sim_M$ is clearly surjective. Using the fact that M is a linear subspace of X, it is easy to check that:

1. if $x \sim_M y$, then $\forall\, \lambda \in \mathbb{K}$ we have $\lambda x \sim_M \lambda y$.
2. if $x \sim_M y$, then $\forall\, z \in X$ we have $x + z \sim_M y + z$.

These two properties guarantee that the following operations are well-defined on X/M:

- vector addition: $\forall\, \phi(x), \phi(y) \in X/M,\ \phi(x) + \phi(y) := \phi(x + y)$
- scalar multiplication: $\forall\, \lambda \in \mathbb{K},\ \forall\, \phi(x) \in X/M,\ \lambda\phi(x) := \phi(\lambda x)$

X/M with the two operations defined above is a vector space and therefore it is often called *quotient vector space*. Then the quotient map ϕ is clearly linear.

Quotient topological vector space

Let X be now a t.v.s. and M a linear subspace of X. Consider the quotient vector space X/M and the quotient map $\phi : X \to X/M$ defined in Section 2.3. Since X is a t.v.s, it is in particular a topological space, so we can consider on X/M the quotient topology defined in Section 2.3. We already know that in this topological setting ϕ is continuous but actually the structure of t.v.s. on X guarantees also that it is open.

Proposition 2.3.2. *For a linear subspace M of a t.v.s. X, the quotient mapping $\phi : X \to X/M$ is open (i.e. carries open sets in X to open sets in X/M) when X/M is endowed with the quotient topology.*

Proof. Let V open in X. Then we have

$$\phi^{-1}(\phi(V)) = V + M = \bigcup_{m \in M} (V + m)$$

Since X is a t.v.s, its topology is translation invariant and so $V + m$ is open for any $m \in M$. Hence, $\phi^{-1}(\phi(V))$ is open in X as union of open sets. By definition, this means that $\phi(V)$ is open in X/M endowed with the quotient topology. □

It is then clear that ϕ carries neighborhoods of a point in X into neighborhoods of a point in X/M and viceversa. Hence, the neighborhoods of the origin in X/M are direct images under ϕ of the neighborhoods of the origin in X. In conclusion, when X is a t.v.s and M is a subspace of X, we can rewrite the definition of quotient topology on X/M in terms of neighborhoods as follows: *the filter of neighborhoods of the origin of X/M is exactly the image under ϕ of the filter of neighborhoods of the origin in X.*

It is not true, in general (not even when X is a t.v.s. and M is a subspace of X), that the quotient map is closed.

Example 2.3.3.
Consider $\mathbb{R}^2$ *with the euclidean topology and the hyperbola* $H := \{(x, y) \in \mathbb{R}^2 : xy = 1\}$*. If* M *is one of the coordinate axes, then* $\mathbb{R}^2/M$ *can be identified with the other coordinate axis and the quotient map* ϕ *with the orthogonal projection on it. All these identifications are also valid for the topologies. The hyperbola* H *is closed in* $\mathbb{R}^2$ *but its image under* ϕ *is the complement of the origin on a straight line which is open.*

Corollary 2.3.4. *For a linear subspace* M *of a t.v.s.* X*, the quotient space* X/M *endowed with the quotient topology is a t.v.s..*

Proof.
For convenience, we denote here by A the vector addition in X/M and just by $+$ the vector addition in X. Let W be a neighbourhood of the origin o in X/M. We aim to prove that $A^{-1}(W)$ is a neighbourhood of (o, o) in $X/M \times X/M$.

The continuity of the quotient map $\phi : X \to X/M$ implies that $\phi^{-1}(W)$ is a neighbourhood of the origin in X. Then, by Theorem 2.1.10-2 (we can apply the theorem because X is a t.v.s.), there exists V neighbourhood of the origin in X s.t. $V + V \subseteq \phi^{-1}(W)$. Hence, by the linearity of ϕ, we get $A(\phi(V) \times \phi(V)) = \phi(V + V) \subseteq W$, i.e. $\phi(V) \times \phi(V) \subseteq A^{-1}(W)$. Since ϕ is also open, $\phi(V)$ is a neighbourhood of the origin o in X/M and so $A^{-1}(W)$ is a neighbourhood of (o, o) in $X/M \times X/M$.

A similar argument gives the continuity of the scalar multiplication. □

Proposition 2.3.5. *Let* X *be a t.v.s. and* M *a linear subspace of* X*. Consider* X/M *endowed with the quotient topology. Then the two following properties are equivalent:*
a) M *is closed*
b) X/M *is Hausdorff*

Proof.
In view of Corollary 2.2.4, (b) is equivalent to say that the complement of the origin in X/M is open w.r.t. the quotient topology. But the complement of the origin in X/M is exactly the image under ϕ of the complement of M in X. Since ϕ is an open continuous map, the image under ϕ of the complement of M in X is open in X/M iff the complement of M in X is open, i.e. (a) holds. □

Corollary 2.3.6. *If* X *is a t.v.s., then* $X/\overline{\{o\}}$ *endowed with the quotient topology is a Hausdorff t.v.s..* $X/\overline{\{o\}}$ *is said to be the* Hausdorff t.v.s. associated *with the t.v.s.* X*. When a t.v.s.* X *is Hausdorff,* X *and* $X/\overline{\{o\}}$ *are topologically isomorphic.*

Proof.
Since X is a t.v.s. and $\{o\}$ is a linear subspace of X, $\overline{\{o\}}$ is a closed linear subspace of X by Proposition 2.1.15-2. Then, by Corollary 2.3.4 and Proposition 2.3.5, $X/\overline{\{o\}}$ is a Hausdorff t.v.s.. If in addition X is Hausdorff, then Corollary 2.2.4 guarantees that $\overline{\{o\}} = \{o\}$ in X. Therefore, the quotient map $\phi : X \to X/\overline{\{o\}}$ is also injective because in this case $Ker(\phi) = \{o\}$. Hence, ϕ is a topological isomorphism (i.e. bijective, continuous, open, linear) between X and $X/\overline{\{o\}}$ which is indeed $X/\{o\}$. □

2.4 Continuous linear mappings between t.v.s.

Let X and Y be two vector spaces over $\mathbb{K}$ and $f : X \to Y$ a linear map. We define the *image* of f, and denote it by $Im(f)$, as the subset of Y:

$$Im(f) := \{y \in Y \,:\, \exists\, x \in X \text{ s.t. } y = f(x)\}.$$

We define the *kernel* of f, and denote it by $Ker(f)$, as the subset of X:

$$Ker(f) := \{x \in X \,:\, f(x) = 0\}.$$

Both $Im(f)$ and $Ker(f)$ are linear subspaces of Y and X, respectively. We have then the diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\phi"] & Im(f) \arrow[r, "i"] & Y \\
X/Ker(f) \arrow[ur, "\bar{f}"] & &
\end{tikzcd}$$

where i is the natural injection of $Im(f)$ into Y, i.e. the mapping which to each element y of $Im(f)$ assigns that same element y regarded as an element of Y; ϕ is the canonical map of X onto its quotient $X/Ker(f)$. The mapping $\bar{f}$ is defined so as to make the diagram commutative, which means that:

$$\forall\, x \in X,\; f(x) = \bar{f}(\phi(x)).$$

Note that

- $\bar{f}$ is well-defined.
  Indeed, if $\phi(x) = \phi(y)$, i.e. $x - y \in Ker(f)$, then $f(x-y) = 0$ that is $f(x) = f(y)$ and so $\bar{f}(\phi(x)) = \bar{f}(\phi(y))$.

- $\bar{f}$ is linear.
  This is an immediate consequence of the linearity of f and of the linear structure of $X/Ker(f)$.
- $\bar{f}$ is a one-to-one map of $X/Ker(f)$ onto $Im(f)$.
  The onto property is evident from the definition of $Im(f)$ and of $\bar{f}$. As for the one-to-one property, note that $\bar{f}(\phi(x)) = \bar{f}(\phi(y))$ means by definition that $f(x) = f(y)$, i.e. $f(x-y) = 0$. This is equivalent, by linearity of f, to say that $x-y \in Ker(f)$, which means that $\phi(x) = \phi(y)$.

The set of all linear maps (continuous or not) of a vector space X into another vector space Y is denoted by $\mathcal{L}(X;Y)$. Note that $\mathcal{L}(X;Y)$ is a vector space for the natural addition and multiplication by scalars of functions. Recall that when $Y = \mathbb{K}$, the space $\mathcal{L}(X;Y)$ is denoted by X^* and it is called the *algebraic dual* of X (see Definition 1.2.4).

Let us not turn to consider linear mapping between two t.v.s. X and Y. Since they posses a topological structure, it is natural to study in this setting continuous linear mappings.

Lemma 2.4.1. *Let $f : X \to Y$ a linear map between two t.v.s. X and Y. If Y is Hausdorff and f is continuous, then $Ker(f)$ is closed in X.*

Proof.
Clearly, $Ker(f) = f^{-1}(\{o\})$. Since Y is a Hausdorff t.v.s., $\{o\}$ is closed in Y and so, by the continuity of f, $Ker(f)$ is also closed in Y. □

Note that $Ker(f)$ might be closed in X also when Y is not Hausdorff. For instance, when $f \equiv 0$ or when f is injective and X is Hausdorff.

Proposition 2.4.2. *Let $f : X \to Y$ a linear map between two t.v.s. X and Y. The map f is continuous if and only if the map $\bar{f}$ is continuous.*

Proof.
Suppose f continuous and let U be an open subset in $Im(f)$ (endowed with the subspace topology induced by the topology on Y). Then $f^{-1}(U)$ is open in X. By definition of $\bar{f}$, we have $\bar{f}^{-1}(U) = \phi(f^{-1}(U))$. Since the quotient map $\phi : X \to X/Ker(f)$ is open, $\phi(f^{-1}(U))$ is open in $X/Ker(f)$. Hence, $\bar{f}^{-1}(U)$ is open in $X/Ker(f)$ and so the map $\bar{f}$ is continuous. Viceversa, suppose that $\bar{f}$ is continuous. Since $f = \bar{f} \circ \phi$ and ϕ is continuous, f is also continuous as composition of continuous maps.

□

In general, the inverse of $\bar{f}$, which is well defined on $Im(f)$ since $\bar{f}$ is injective, is not continuous. In other words, $\bar{f}$ is not necessarily bi-continuous.

The set of all continuous linear maps of a t.v.s. X into another t.v.s. Y is denoted by $L(X;Y)$ and it is a vector subspace of $\mathcal{L}(X;Y)$. When $Y = \mathbb{K}$, the space $L(X;Y)$ is usually denoted by X' which is called the *topological dual* of X, in order to underline the difference with X^* the algebraic dual of X. X' is a vector subspace of X^* and is exactly the vector space of all continuous linear functionals, or continuous linear forms, on X. The vector spaces X' and $L(X;Y)$ will play an important role in the forthcoming and will be equipped with various topologies.