In general, the inverse of $\bar{f}$, which is well defined on $Im(f)$ since $\bar{f}$ is injective, is not continuous. In other words, $\bar{f}$ is not necessarily bi-continuous.

The set of all continuous linear maps of a t.v.s. X into another t.v.s. Y is denoted by $L(X;Y)$ and it is a vector subspace of $\mathcal{L}(X;Y)$. When $Y = \mathbb{K}$, the space $L(X;Y)$ is usually denoted by X' which is called the *topological dual* of X, in order to underline the difference with X^* the algebraic dual of X. X' is a vector subspace of X^* and is exactly the vector space of all continuous linear functionals, or continuous linear forms, on X. The vector spaces X' and $L(X;Y)$ will play an important role in the forthcoming and will be equipped with various topologies.

2.5 Completeness for t.v.s.

This section aims to treat completeness for most general types of topological vector spaces, beyond the traditional metric framework. As well as in the case of metric spaces, we need to introduce the definition of a Cauchy sequence in a t.v.s..

Definition 2.5.1. *A sequence $S := \{x_n\}_{n\in\mathbb{N}}$ of points in a t.v.s. X is said to be a* Cauchy sequence *if*

$$\forall\, U \in \mathcal{F}(o) \, in \, X,\ \exists\, N \in \mathbb{N} :\ x_m - x_n \in U,\ \forall m, n \geq N. \tag{2.2}$$

This definition agrees with the usual one if the topology of X is defined by a translation-invariant metric d. Indeed, in this case, a basis of neighbourhoods of the origin is given by all the open balls centered at the origin. Therefore, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in such (X, d) iff $\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{N}$: $x_m - x_n \in B_\varepsilon(o)$, $\forall m, n \geq N$, i.e. $d(x_m, x_n) = d(x_m - x_n, o) < \varepsilon$.

By using the subsequences $S_m := \{x_n \in S\ :\ n \geq m\}$ of S, we can easily rewrite (2.2) in the following way

$$\forall\, U \in \mathcal{F}(o) \text{ in } X,\ \exists\, N \in \mathbb{N} :\ S_N - S_N \subset U.$$

As we have already observed in Chapter 1, the collection $\mathcal{B} := \{S_m\ :\ m \in \mathbb{N}\}$ is a basis of the filter $\mathcal{F}_S$ associated with the sequence S. This immediately suggests what the definition of a Cauchy filter should be:

Definition 2.5.2. *A filter $\mathcal{F}$ on a subset A of a t.v.s. X is said to be a* Cauchy filter *if*

$$\forall\, U \in \mathcal{F}(o) \, in \, X,\ \exists\, M \subset A :\ M \in \mathcal{F} \text{ and } M - M \subset U.$$

In order to better illustrate this definition, let us come back to our reference example of a t.v.s. X whose topology is defined by a translation-invariant metric d. For any subset M of (X, d), recall that the diameter of M is defined as $diam(M) := \sup_{x,y \in M} d(x, y)$. Now if $\mathcal{F}$ is a Cauchy filter on X then, by definition, for any $\varepsilon > 0$ there exists $M \in \mathcal{F}$ s.t. $M - M \subset B_\varepsilon(o)$ and this simply means that $diam(M) \leq \varepsilon$. Therefore, Definition 2.5.2 can be rephrased in this case as follows:
a filter $\mathcal{F}$ on a subset A of such a metric t.v.s. X is a Cauchy filter if it contains subsets of A of arbitrarily small diameter.

Going back to the general case, the following statement clearly holds.

Proposition 2.5.3.
The filter associated with a Cauchy sequence in a t.v.s. X is a Cauchy filter.

Proposition 2.5.4.
Let X be a t.v.s.. Then the following properties hold:
a) The filter of neighborhoods of a point $x \in X$ is a Cauchy filter on X.
b) A filter finer than a Cauchy filter is a Cauchy filter.
c) Every converging filter is a Cauchy filter.

Proof.
a) Let $\mathcal{F}(x)$ be the filter of neighborhoods of a point $x \in X$ and let $U \in \mathcal{F}(o)$. By Theorem 2.1.10, there exists $V \in \mathcal{F}(o)$ such that $V - V \subset U$ and so such that $(V + x) - (V + x) \subset U$. Since X is a t.v.s., we know that $\mathcal{F}(x) = \mathcal{F}(o) + x$ and so $M := V + x \in \mathcal{F}(x)$. Hence, we have proved that for any $U \in \mathcal{F}(o)$ there exists $M \in \mathcal{F}(x)$ s.t. $M - M \subset U$, i.e. $\mathcal{F}(x)$ is a Cauchy filter.
b) Let $\mathcal{F}$ and $\mathcal{F}'$ be two filters of subsets of X such that $\mathcal{F}$ is a Cauchy filter and $\mathcal{F} \subseteq \mathcal{F}'$. Since $\mathcal{F}$ is a Cauchy filter, by Definition 2.5.2, for any $U \in \mathcal{F}(o)$ there exists $M \in \mathcal{F}$ s.t. $M - M \subset U$. But $\mathcal{F}'$ is finer than $\mathcal{F}$, so M belongs also to $\mathcal{F}'$. Hence, $\mathcal{F}'$ is obviously a Cauchy filter.
c) If a filter $\mathcal{F}$ converges to a point $x \in X$ then $\mathcal{F}(x) \subseteq \mathcal{F}$ (see Definition 1.1.28). By a), $\mathcal{F}(x)$ is a Cauchy filter and so b) implies that $\mathcal{F}$ itself is a Cauchy filter.

□

The converse of c) is in general false, in other words not every Cauchy filter converges.

Definition 2.5.5.
A subset A of a t.v.s. X is said to be complete *if every Cauchy filter on A converges to a point x of A.*

It is important to distinguish between completeness and sequentially completeness.

Definition 2.5.6.
A subset A of a t.v.s. X is said to be sequentially complete *if any Cauchy sequence in A converges to a point in A.*

It is not hard to prove that complete always implies sequentially complete. The converse is in general false (see Example 2.5.9). We will encounter an important class of t.v.s. for which the two notions coincide (see Sheet 6, Exercise 3-a)).

Proposition 2.5.7.
If a subset A of a t.v.s. X is complete then A is sequentially complete.

Proof.
Let $S := \{x_n\}_{n\in\mathbb{N}}$ a Cauchy sequence of points in A. Then Proposition 2.5.3 guarantees that the filter $\mathcal{F}_S$ associated to S is a Cauchy filter in A. By the completeness of A we get that there exists $x \in A$ such that $\mathcal{F}_S$ converges to x. This is equivalent to say that the sequence S is convergent to $x \in A$ (see Proposition 1.1.30). Hence, A is sequentially complete.

□

Before showing an example of a subset of a t.v.s. which is sequentially complete but not complete, let us introduce two useful properties about completeness in t.v.s..

Proposition 2.5.8.
a) In a Hausdorff t.v.s. X, any complete subset is closed.
b) In a complete t.v.s. X, any closed subset is complete.

Example 2.5.9.
Let $X := \prod_{i\in J}\mathbb{R}$ with $|J| > \aleph_0$ endowed with the product topology given by considering each copy of $\mathbb{R}$ equipped with the usual topology given by the modulus. Note that X is a Hausdorff t.v.s. as it is product of Hausdorff t.v.s. (see Sheet 4, Exercise 3). Denote by H the subset of X consisting of all vectors $\underline{x} = (x_i)_{i\in J}$ in X with only countably many non-zero coordinates x_i.

<u>*Claim:*</u> *H is sequentially complete but not complete.*

Proof. of Claim.
Let us first make some observations on H.

- H is strictly contained in X.
  Indeed, any vector $\underline{y} \in X$ with all non-zero coordinates does not belong to H because $|J| > \aleph_0$.
- H is dense in X.
  In fact, let $\underline{x} = (x_i)_{i \in J} \in X$ and U a neighbourhood of $\underline{x}$ in X. Then, by definition of product topology on X, there exist $U_i \subseteq \mathbb{R}$ s.t $\prod_{i \in J} U_i \subseteq U$ and U_i is a neighbourhood of x_i in $\mathbb{R}$ for all $i \in J$ with $U_i \neq \mathbb{R}$ for all $i \in I$ where $I \subset J$ s.t. $|I| < \infty$. Take $\underline{y} := (y_i)_{i \in J}$ s.t. $y_i \in U_i$ for all $i \in J$ with $y_i \neq 0$ for all $i \in I$ and $y_i = 0$ otherwise. Then clearly $\underline{y} \in U$ but also $\underline{y} \in H$ because it has only finitely many non-zero coordinates. Hence, $U \cap H \neq \emptyset$ and so $\overline{H} = X$.

Now suppose that H is complete, then by Proposition 2.5.8-a) we have that H is closed. Therefore, by the density of H in X, it follows that $H = \overline{H} = X$ which contradicts the first of the property above. Hence, H is not complete.

In the end, let us show that H is sequentially complete. Let $(\underline{x}_n)_{n \in \mathbb{N}}$ a Cauchy sequence of vectors $\underline{x}_n = (x_n^{(i)})_{i \in J}$ in H. Then for each $i \in J$ we have that the sequence of the $i-th$ coordinates $(x_n^{(i)})_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. By the completeness (i.e. the sequentially completeness) of $\mathbb{R}$ we have that for each $i \in J$, the sequence $(x_n^{(i)})_{n \in \mathbb{N}}$ converges to a point $x^{(i)} \in \mathbb{R}$. Set $\underline{x} := (x^{(i)})_{i \in J}$. Then:

- $\underline{x} \in H$, because for each $n \in \mathbb{N}$ only countably many $x_n^{(i)} \neq 0$ and so only countably many $x^{(i)} \neq 0$.
- the sequence $(\underline{x}_n)_{n \in \mathbb{N}}$ converges to $\underline{x}$ in H. In fact, for any U neighbourhood of $\underline{x}$ in X there exist $U_i \subseteq \mathbb{R}$ s.t $\prod_{i \in J} U_i \subseteq U$ and U_i is a neighbourhood of x_i in $\mathbb{R}$ for all $i \in J$ with $U_i \neq \mathbb{R}$ for all $i \in I$ where $I \subset J$ s.t. $|I| < \infty$. Since for each $i \in J$, the sequence $(x_n^{(i)})_{n \in \mathbb{N}}$ converges to $x^{(i)}$ in $\mathbb{R}$, we get that for each $i \in J$ there exists $N_i \in \mathbb{N}$ s.t. $x_n^{(i)} \in U_i$ for all $n \geq N_i$. Take $N := \max_{i \in I} N_i$ (the max exists because I is finite). Then for each $i \in J$ we get $x_n^{(i)} \in U_i$ for all $n \geq N$, i.e. $\underline{x}_n \in U$ for all $n \geq N$ which proves the convergence of $(\underline{x}_n)_{n \in \mathbb{N}}$ to $\underline{x}$.

Hence, we have showed that every Cauchy sequence in H is convergent.

□

In order to prove Proposition 2.5.8, we need two small lemmas regarding convergence of filters in a topological space.

Lemma 2.5.10. *Let $\mathcal{F}$ be a filter of a topological Hausdorff space X. If $\mathcal{F}$ converges to $x \in X$ and also to $y \in X$, then $x = y$.*

Proof. (Sheet 6, Exercise 1) □

Lemma 2.5.11. *Let A be a subset of a topological space X. Then $x \in \overline{A}$ if and only if there exists a filter $\mathcal{F}$ of subsets of X such that $A \in \mathcal{F}$ and $\mathcal{F}$ converges to x.*

Proof. (Sheet 6, Exercise 2) □

Proof. of Proposition 2.5.8

a) Let A be a complete subset of a Hausdorff t.v.s. X and let $x \in \overline{A}$. By Lemma 2.5.11, $x \in \overline{A}$ implies that there exists a filter $\mathcal{F}$ of subsets of X s.t. $A \in \mathcal{F}$ and $\mathcal{F}$ converges to x. Therefore, by Proposition 2.5.4-c), $\mathcal{F}$ is a Cauchy filter. Consider now $\mathcal{F}_A := \{U \in \mathcal{F} \,:\, U \subseteq A\} \subset \mathcal{F}$. It is easy to see that $\mathcal{F}_A$ is a Cauchy filter on A and so the completeness of A ensures that $\mathcal{F}_A$ converges to a point $y \in A$. Hence, any nbhood V of y in A belongs to $\mathcal{F}_A$ and so to $\mathcal{F}$. By definition of subset topology, this means that for any nbhood U of y in X we have $U \cap A \in \mathcal{F}$ and so $U \in \mathcal{F}$ (since $\mathcal{F}$ is a filter). Then $\mathcal{F}$ converges to y. Since X is Hausdorff, Lemma 2.5.10 establishes the uniqueness of the limit point of $\mathcal{F}$, i.e. $x = y$ and so $\overline{A} = A$.

b) Let A be a closed subset of a complete t.v.s. X and let $\mathcal{F}_A$ be any Cauchy filter on A. Take the filter $\mathcal{F} := \{F \subseteq X | B \subseteq F \text{ for some } B \in \mathcal{F}_A\}$. It is clear that $\mathcal{F}$ contains A and is finer than the Cauchy filter $\mathcal{F}_A$. Therefore, by Proposition 2.5.4-b), $\mathcal{F}$ is also a Cauchy filter. Then the completeness of the t.v.s. X gives that $\mathcal{F}$ converges to a point $x \in X$, i.e. $\mathcal{F}(x) \subseteq \mathcal{F}$. By Lemma 2.5.11, this implies that actually $x \in \overline{A}$ and, since A is closed, that $x \in A$. Now any neighbourhood of $x \in A$ in the subset topology is of the form $U \cap A$ with $U \in \mathcal{F}(x)$. Since $\mathcal{F}(x) \subseteq \mathcal{F}$ and $A \in \mathcal{F}$, we have $U \cap A \in \mathcal{F}$. Therefore, there exists $B \in \mathcal{F}_A$ s.t. $B \subseteq U \cap A \subset A$ and so $U \cap A \in \mathcal{F}_A$. Hence, $\mathcal{F}_A$ converges $x \in A$, i.e. A is complete.

□

When a t.v.s. is not complete, it makes sense to ask if it is possible to embed it in a complete one. The following theorem establishes a positive answer to this question and the proof (see [1, Section 2.5, pp. 37–42], [2, Section 5, 41–48]) provides an abstract procedure for associating to an arbitrary Hausdorff t.v.s. X a complete Hausdorff t.v.s. $\hat{X}$ called the *completion* of X.

Theorem 2.5.12.
Let X be a Haudorff t.v.s.. Then there exists a complete Hausdorff t.v.s. $\hat{X}$ and a mapping $i : X \to \hat{X}$ with the following properties:
a) The mapping i is a topological monomorphism.
b) The image of X under i is dense in $\hat{X}$.
c) For every complete Hausdorff t.v.s. Y and for every continuous linear map $f : X \to Y$, there is a continuous linear map $\hat{f} : \hat{X} \to Y$ such that the following diagram is commutative:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"] & Y \\ \hat{X} \arrow[ur, "\hat{f}"'] & \end{tikzcd}$$

Furthermore:

I) Any other pair $(\hat{X}_1, i_1)$, consisting of a complete Hausdorff t.v.s. $\hat{X}_1$ and of a mapping $i_1 : X \to \hat{X}_1$ such that properties (a) and (b) hold substituting $\hat{X}$ with $\hat{X}_1$ and i with i_1, is topologically isomorphic to $(\hat{X}, i)$. This means that there is a topological isomorphism j of $\hat{X}$ onto $\hat{X}_1$ such that the following diagram is commutative:

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[d, "i"] & \hat{X}_1 \\ \hat{X} \arrow[ur, "j"'] & \end{tikzcd}$$

II) Given Y and f as in property (c), the continuous linear map $\hat{f}$ is unique.

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