In particular, the collection $\mathcal{M}$ of all multiples ρU of an absorbing absolutely convex subset U of a vector space X is a basis of neighborhoods of the origin for a locally convex topology on X compatible with the linear structure (this ceases to be true, in general, if we relax the conditions on U).

Proof. First of all, let us observe that for any $\rho \in \mathbb{K}$, we have that ρU is absorbing and absolutely convex since U has such properties.

For any $A, B \in \mathcal{M}$, there exist $\lambda, \mu \in \mathbb{K}$ s.t. $A = \lambda U$ and $B = \mu U$. W.l.o.g. we can assume $|\lambda| \leq |\mu|$ and so $\frac{\lambda}{\mu} U \subseteq U$, i.e. $A \subseteq B$. Hence, a) and b) in Theorem 4.1.14 are fulfilled since $A \cap B = A \in \mathcal{M}$ and, for any $\rho \in \mathbb{K}$, $\rho A = \rho\lambda U \in \mathcal{M}$.

Therefore, Theorem 4.1.14 ensures that $\mathcal{M}$ is a basis of neighbourhoods of the origin of a topology which makes X into a l.c. t.v.s..

□

4.2 Connection to seminorms

In applications it is often useful to define a locally convex space by means of a system of seminorms. In this section we will investigate the relation between locally convex t.v.s. and seminorms.

Definition 4.2.1. *Let X be a vector space. A function $p : X \to \mathbb{R}$ is called a* seminorm *if it satisfies the following conditions:*

1. *p is subadditive: $\forall x, y \in X$, $p(x + y) \leq p(x) + p(y)$.*
2. *p is positively homogeneous: $\forall x, y \in X$, $\forall \lambda \in \mathbb{K}, p(\lambda x) = |\lambda| p(x)$.*

Definition 4.2.2.
A seminorm p on a vector space X is a norm *if $p^{-1}(\{0\}) = \{o\}$ (i.e. if $p(x) = 0$ implies $x = o$).*

Proposition 4.2.3. *Let p be a seminorm on a vector space X. Then the following properties hold:*

- *p is symmetric.*
- *$p(o) = 0$.*
- *$|p(x) - p(y)| \leq p(x - y)$, $\forall\, x, y \in X$.*
- *$p(x) \geq 0$, $\forall\, x \in X$.*
- *$Ker(p)$ is a linear subspace.*

Proof.

- The symmetry of p directly follows from the positive homogeneity of p. Indeed, for any $x \in X$ we have

$$p(-x) = p(-1 \cdot x) = |-1| p(x) = p(x).$$

- Using again the positive homogeneity of p we get that $p(o) = p(0 \cdot x) = 0 \cdot p(x) = 0$.
- For any $x, y \in X$, the subadditivity of p guarantees the following inequalities:

  $$p(x) = p(x-y+y) \leq p(x-y)+p(y) \quad \text{and} \quad p(y) = p(y-x+x) \leq p(y-x)+p(x)$$

  which establish the third property.
- The previous property directly gives the nonnegativity of p. In fact, for any $x \in X$ we get

  $$0 \leq |p(x) - p(o)| \leq p(x - o) = p(x).$$
- Let $x, y \in Ker(p)$ and $\alpha, \beta \in \mathbb{K}$. Then

  $$p(\alpha x + \beta y) \leq |\alpha| p(x) + |\beta| p(y) = 0$$

  which implies,by the nonnegativity of p, that $p(\alpha x + \beta y) = 0$. Hence, we have $\alpha x + \beta y \in Ker(p)$.

$\square$

Examples 4.2.4.

a) *Suppose $X = \mathbb{R}^n$ and let M be a vector subspace of X. Set for any $x \in X$*

$$p_M(x) := \inf_{m \in M} \|x - m\|$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$, i.e. $p_M(x)$ is the distance from the point x to M in the usual sense. If $dim(M) \geq 1$ then p_M is a seminorm and not a norm (M is exactly the kernel of p_M). When $M = \{o\}$, $p_M(\cdot) = \|\cdot\|$.

b) *Let $\mathcal{C}(\mathbb{R})$ be the vector space of all real valued continuous functions on the real line. For any bounded interval $[a, b]$ with $a, b \in \mathbb{R}$ and $a < b$, we define for any $f \in \mathcal{C}(\mathbb{R})$:*

$$p_{[a,b]}(f) := \sup_{a \leq t \leq b} |f(t)|.$$

$p_{[a,b]}$ is a seminorm but is never a norm because it might be that $f(t) = 0$ for all $t \in [a, b]$ (and so that $p_{[a,b]}(f) = 0$) but $f \not\equiv 0$. Other seminorms are the following ones:

$$q(f) := |f(0)| \quad \text{and} \quad q_p(f) := \left(\int_a^b |f(t)|^p dt \right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty.$$

Note that if $0 < p < 1$ then q_p is not subadditive and so it is not a seminorm (see Sheet 8, Exercise 3).

c) Let X be a vector space on which is defined a nonnegative sesquilinear Hermitian form $B : X \times X \to \mathbb{K}$. Then the function

$$p_B(x) := B(x,x)^{\frac{1}{2}}$$

is a seminorm. p_B is a norm if and only if B is positive definite (i.e. $B(x,x) > 0$, $\forall x \neq o$).

Seminorms on vector spaces are strongly related to a special kind of functionals, i.e. *Minkowski functionals*. Let us investigate more in details such a relation. Note that we are still in the realm of vector spaces with no topology!

Definition 4.2.5. *Let X be a vector space and A a non-empty subset of X. We define the* Minkowski functional (or gauge) *of A to be the mapping:*

$$\begin{array}{rccl} p_A : & X & \to & \mathbb{R} \\ & x & \mapsto & p_A(x) := \inf\{\lambda > 0 : x \in \lambda A\} \end{array}$$

(where $p_A(x) = \infty$ if the set $\{\lambda > 0 : x \in \lambda A\}$ is empty).

It is then natural to ask whether there exists a class of subsets for which the associated Minkowski functionals are actually seminorms. The answer is positive for a class of subsets which we have already encountered in the previous section, namely for absorbing absolutely convex subsets. Actually we have even more as established in the following lemma.

Notation 4.2.6. *Let X be a vector space and p a seminorm on X. The sets*

$$\mathring{U}_p = \{x \in X : p(x) < 1\} \text{ and } U_p = \{x \in X : p(x) \leq 1\}.$$

are said to be, respectively, the closed and the open unit semiball *of p.*

Lemma 4.2.7. *Let X be a vector space. If A is a non-empty subset of X which is absorbing and absolutely convex, then the associated Minkowski functional p_A is a seminorm and $\mathring{U}_{p_A} \subseteq A \subseteq U_{p_A}$. Conversely, if q is a seminorm on X then $\mathring{U}_q$ is an absorbing absolutely convex set and $q = p_{\mathring{U}_q}$.*

Proof. Let A be a non-empty subset of X which is absorbing and absolutely convex and denote by p_A the associated Minkowski functional. We want to show that p_A is a seminorm.

- First of all, note that $p_A(x) < \infty$ for all $x \in X$ because A is absorbing. Indeed, by definition of absorbing set, for any $x \in X$ there exists $\rho_x > 0$ s.t. for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq \rho_x$ we have $\lambda x \in A$ and so the set $\{\lambda > 0 : x \in \lambda A\}$ is never empty i.e. p_A has only finite nonnegative values. Moreover, since $o \in A$, we also have that $o \in \lambda A$ for any $\lambda \in \mathbb{K}$ and so $p_A(o) = \inf\{\lambda > 0 : o \in \lambda A\} = 0$.
- The balancedness of A implies that p_A is positively homogeneous. Since we have already showed that $p_A(o) = 0$ it remains to prove the positive homogeneity of p_A for non-null scalars. Since A is balanced we have that for any $x \in X$ and for any $\xi, \lambda \in \mathbb{K}$ with $\xi \neq 0$ the following holds:

$$\xi x \in \lambda A \text{ if and only if } x \in \frac{\lambda}{|\xi|}A. \tag{4.1}$$

  Indeed, A balanced guarantees that $\xi A = |\xi|A$ and so $x \in \frac{\lambda}{|\xi|}A$ is equivalent to $\xi x \in \lambda \frac{\xi}{|\xi|}A = \lambda A$. Using (4.1), we get that for any $x \in X$ and for any $\xi \in \mathbb{K}$ with $\xi \neq 0$:

$$\begin{aligned} p_A(\xi x) &= \inf\{\lambda > 0 : \xi x \in \lambda A\} \\ &= \inf\left\{\lambda > 0 : x \in \frac{\lambda}{|\xi|}A\right\} \\ &= \inf\left\{|\xi|\frac{\lambda}{|\xi|} > 0 : x \in \frac{\lambda}{|\xi|}A\right\} \\ &= |\xi| \inf\{\mu > 0 : x \in \mu A\} = |\xi| p_A(x). \end{aligned}$$

- The convexity of A ensures the subadditivity of p_A. Take $x, y \in X$. By definition of Minkowski functional, for every $\varepsilon > 0$ there exists $\lambda, \mu > 0$ s.t.

$$\lambda \leq p_A(x) + \varepsilon \text{ and } x \in \lambda A$$

  and

$$\mu \leq p_A(y) + \varepsilon \text{ and } y \in \mu A.$$

  Then, by the convexity of A, we obtain that $\frac{\lambda}{\lambda+\mu}A + \frac{\mu}{\lambda+\mu}A \subseteq A$, i.e. $\lambda A + \mu A \subseteq (\lambda + \mu)A$, and therefore $x + y \in (\lambda + \mu)A$. Hence:

$$p_A(x + y) = \inf\{\delta > 0 : x + y \in \delta A\} \leq \lambda + \mu \leq p_A(x) + p_A(y) + 2\varepsilon$$

  which proves the subadditivity of p_A since ε is arbitrary.

We can then conclude that p_A is a seminorm. Furthermore, we have the following inclusions:

$$\mathring{U}_{p_A} \subseteq A \subseteq U_{p_A}.$$

In fact, if $x \in \mathring{U}_{p_A}$ then $p_A(x) < 1$ and so there exists $0 \leq \lambda < 1$ s.t. $x \in \lambda A$. Since A is balanced, for such λ we have $\lambda A \subseteq A$ and therefore $x \in A$. On the other hand, if $x \in A$ then clearly $1 \in \{\lambda > 0 : x \in \lambda A\}$ which gives $p_A(x) \leq 1$ and so $x \in U_{p_A}$.

Conversely, let us take any seminorm q on X. Let us first show that $\mathring{U}_q$ is absorbing and absolutely convex and then that q coincides with the Minkowski functional associated to $\mathring{U}_q$.

- $\mathring{U}_q$ is absorbing.
  Let x be any point in X. If $q(x) = 0$ then clearly $x \in \mathring{U}_q$. If $q(x) > 0$, we can take $0 < \rho < \frac{1}{q(x)}$ and then for any $\lambda \in \mathbb{K}$ s.t. $|\lambda| \leq \rho$ the positive homogeneity of q implies that $q(\lambda x) = |\lambda| q(x) \leq \rho q(x) < 1$, i.e. $\lambda x \in \mathring{U}_q$.
- $\mathring{U}_q$ is balanced.
  For any $x \in \mathring{U}_q$ and for any $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$, again by the positive homogeneity of q, we get: $q(\lambda x) = |\lambda| q(x) \leq q(x) < 1$ i.e. $\lambda x \in \mathring{U}_q$.
- $\mathring{U}_q$ is convex.
  For any $x, y \in \mathring{U}_q$ and any $t \in [0, 1]$, by both the properties of seminorm, we have that $q(tx + (1-t)y) \leq tq(x) + (1-t)q(y) < t + 1 - t = 1$ i.e. $tx + (1-t)y \in \mathring{U}_q$.

Moreover, for any $x \in X$ we easily see that

$$p_{\mathring{U}_q}(x) = \inf\{\lambda > 0 : x \in \lambda \mathring{U}_q\} = \inf\{\lambda > 0 : q(x) < \lambda\} = q(x).$$

□

We are now ready to see the connection between seminorms and locally convex t.v.s..

Definition 4.2.8. *Let X be a vector space and $\mathcal{P} := \{p_i\}_{i \in I}$ a family of seminorms on X. The coarsest topology $\tau_{\mathcal{P}}$ on X s.t. each p_i is continuous is said to be the* topology induced or generated by the family of seminorms $\mathcal{P}$.

Theorem 4.2.9. *Let X be a vector space and $\mathcal{P} := \{p_i\}_{i \in I}$ a family of seminorms. Then the topology induced by the family $\mathcal{P}$ is the unique topology making X into a locally convex t.v.s. and having as a basis of neighbourhoods of the origin in X the following collection:*

$$\mathcal{B} := \left\{ \{x \in X : p_{i_1}(x) < \varepsilon, \ldots, p_{i_n}(x) < \varepsilon\} : i_1, \ldots, i_n \in I, n \in \mathbb{N}, \varepsilon > 0, \epsilon \in \mathbb{R} \right\}.$$

Viceversa, the topology of an arbitrary locally convex t.v.s. is always induced by a family of seminorms (often called generating*).*

Proof.
Let us first show that the collection $\mathcal{B}$ is a basis of neighbourhoods of the origin for the unique topology τ making X into a locally convex t.v.s. by using Theorem 4.1.14 and then let us prove that τ actually coincides with the topology induced by the family $\mathcal{P}$.

For any $i \in I$ and any $\varepsilon > 0$, consider the set $\{x \in X : p_i(x) < \varepsilon\} = \varepsilon \mathring{U}_{p_i}$. This is absorbing and absolutely convex, since we have already showed above that $\mathring{U}_{p_i}$ fulfills such properties. Therefore, any element of $\mathcal{B}$ is an absorbing absolutely convex subset of X as finite intersection of absorbing absolutely convex sets. Moreover, both properties a) and b) of Theorem 4.1.14 are clearly satisfied by $\mathcal{B}$. Hence, Theorem 4.1.14 guarantees that there exists a unique topology τ on X s.t. (X, τ) is a locally convex t.v.s. and $\mathcal{B}$ is a basis of neighbourhoods of the origin for τ.

Let us consider (X, τ). Then for any $i \in I$, the seminorm p_i is continuous, because for any $\varepsilon > 0$ we have $p_i^{-1}([0, \varepsilon[) = \{x \in X : p_i(x) < \varepsilon\} \in \mathcal{B}$ which means that $p_i^{-1}([0, \varepsilon[)$ is a neighbourhood of the origin in (X, τ). Therefore, the topology $\tau_{\mathcal{P}}$ induced by the family $\mathcal{P}$ is by definition coarser than τ. On the other hand, each p_i is also continuous w.r.t. $\tau_{\mathcal{P}}$ and so $\mathcal{B} \subseteq \tau_{\mathcal{P}}$. But $\mathcal{B}$ is a basis for τ, then necessarily τ is coarser than $\tau_{\mathcal{P}}$. Hence, $\tau \equiv \tau_{\mathcal{P}}$.