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TOPOLOGICAL VECTOR SPACES–SS 2017

Exercise Sheet 1

This assignment is due by Friday the 5th of May by 11:45. Please, hand in your solutions in postbox 15 near F411.

1) a) Prove the following characterization of basis of a topological space.

Proposition 1. *Let X be a set and let $\mathcal{B}$ be a collection of subsets of X. $\mathcal{B}$ is a basis for a topology τ on X iff*

i. $\mathcal{B}$ covers X, i.e. $\forall x \in X$, $\exists B \in \mathcal{B}$ s.t. $x \in B$. In other words, $X = \cup_{B\in\mathcal{B}} B$.

ii. If $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

b) Let $\mathcal{B}$ be the collection of all intervals (a, b) in $\mathbb{R}$ together with all the sets of the form $(a, b) - K$, where $K := \{\frac{1}{n} : n \in \mathbb{N}\}$. Prove that $\mathcal{B}$ is the basis for a topology on $\mathbb{R}$, which is usually called the $K-$*topology* on $\mathbb{R}$.

2) Show the following statements.

a) The family $\mathcal{G}$ of all subsets of a set X containing a fixed non-empty subset A is a filter and $\mathcal{B} = \{A\}$ is its base. $\mathcal{G}$ is known as the *principal filter* generated by A.

b) Given a topological space X and $x \in X$, the family $\mathcal{F}(x)$ of all neighbourhoods of x is a filter.

c) Let $S := \{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in a set X. Then the family $\mathcal{F} := \{A \subset X : |S\backslash A| < \infty\}$ is a filter and it is known as the *filter associated* to the sequence S. For each $m \in \mathbb{N}$, set $S_m := \{x_n \in S : n \geq m\}$. Then $\mathcal{B} := \{S_m : m \in \mathbb{N}\}$ is a basis for $\mathcal{F}$.

3) Establish which of the following topologies on $\mathbb{R}$ are comparable and for each comparable pair say which one is finer.

- τ_1 :=standard topology, whose basis is $\mathcal{B}_1 := \{(a, b) : a, b \in \mathbb{R} \text{ with } a < b\}$
- $\tau_2 := K-$topology, whose basis $\mathcal{B}_2$ is the one defined in Exercise 1 b)
- $\tau_3 :=$ lower limit topology, whose basis is $\mathcal{B}_3 := \{[a, b) : a, b \in \mathbb{R} \text{ with } a < b\}$