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Fachbereich Mathematik und Statistik
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TOPOLOGICAL VECTOR SPACES–SS 2017
Exercise Sheet 10

This assignment is due by Tuesday the 18th of July by 13:30 and will be discussed in the tutorial next Friday the 21st of July at 10:00 in F426. Please, hand in your solutions in postbox 15 near F411.

1) Provide an alternative proof of the following result (Proposition 4.4.2 in the lecture notes) without using Theorem 4.1.14 but exploiting Proposition 4.4.1 and the results in Section 4.2.

Proposition. *The collection of all absorbing absolutely convex sets of a vector space X is a basis of neighbourhoods of the origin for the finest locally convex topology on X.*

2) Let $(\mathcal{C}^\infty(\mathbb{R}^d), \tau_\mathcal{P})$ be the space of infinitely differentiable functions endowed with the topology induced by the family $\mathcal{P}$ of semi-norms defined in Exercise Sheet 9, Exercise 2). Use the following theorem (which will be proved in TVS-II) to show that there exists a metric on $\mathcal{C}^\infty(\mathbb{R}^d)$ that induces the topology $\tau_\mathcal{P}$.

Theorem. *Let (X, τ) be a t.v.s.. Then τ is induced by a metric on X if and only if (X, τ) is Hausdorff and there exists a countable basis of neighbourhoods of the origin w.r.t. τ.*

3) Keeping in mind the definition of finite topology on a countable dimensional vector space (see Definition 4.5.1 in the lecture notes), prove the following statements.

a) Let X, Y be two infinite dimensional vector spaces of countable dimension each endowed with the corresponding finite topology. Then the finite topology on the product $X \times Y$ coincides with the product topology.

b) Let X be an infinite dimensional vector space with basis $\{x_n\}_{n\in\mathbb{N}}$ endowed with the finite topology τ_f and (Y, τ) any other topological space. For any $i \in \mathbb{N}$ set $X_i := span\{x_1, \dots, x_i\}$ so that $X = \bigcup_{i=1}^\infty X_i$. A map $f : X \to Y$ is continuous (w.r.t. τ_f and τ) if and only if for each $i \in \mathbb{N}$ the restriction $f|_{X_i}$ of f to X_i is continuous (w.r.t. the euclidean topology and τ).

c) Any countable dimensional vector space endowed with the finite topology is a t.v.s..
(*Hint*: use the properties (a) and (b))