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TOPOLOGICAL VECTOR SPACES–SS 2017

Bonus Sheet

This sheet aims to self-assess your progress and to explicitly work out more details of some of the results proposed in the lectures. You do not need to hand in solutions for these exercises however, you may prepare them by Tuesday July 25th at 13:30, when they will be discussed in D404, in order to score bonus points!

1. Let $\mathbb{R}[x]$ denote the vector space of polynomials in the variable x and endow it with the finite topology τ_{fin}. Consider the subset $T := \{p \in \mathbb{R}[x] : p(x) \geq 0 \text{ for all } x \in \mathbb{R}\}$.
   (a) Show that T is a convex cone and that it is closed w.r.t. τ_{fin}.
   (b) Show that for any $p \in \mathbb{R}[x]$ such that $p(x) < 0$ for some $x \in \mathbb{R}$, there exists a hyperplane H such that T lies in the half-space determined by H that does not contain p.

2. Show that if X and Y are two locally convex t.v.s. whose topologies are respectively generated by the families $\mathcal{P}$ and $\mathcal{Q}$ of seminorms on X, then: $f : X \to Y$ is continuous iff
$$\forall q \in \mathbb{Q}, \exists n \in \mathbb{N}, \exists p_1, \dots, p_n \in \mathcal{P}, \exists C > 0 \text{ s.t. } q(L(x)) \leq C \max_{i=1,\dots,n} p_i(x), \forall x \in X.$$
   (This result is referred as Theorem 4.6.3 in the lecture notes).

3. Let $\mathbb{R}[x]$ denote the vector space of polynomials in the variable x. Consider the subset $C := \{p \in \mathbb{R}[x] : p = 0 \text{ or } p(x) = \sum_{i=0}^{d} a_i x^i \text{ for some } d \in \mathbb{N}_0, a_0, \dots, a_d \in \mathbb{R} \text{ and } a_d > 0\}$.
   (a) Show that C is a convex cone , that $C \cap (-C) = \{0\}$, and that $C \cup (-C) = \mathbb{R}[x]$.
   (b) Show that there is no affine hyperplane H such that C is contained only in one of the two half-spaces determined by H.