Universität Konstanz
Fachbereich Mathematik und Statistik
Dr. Maria Infusino
Patrick Michalski

TOPOLOGICAL VECTOR SPACES–SS 2017

Exercise Sheet 2

This assignment is due by Friday the 12th of May by 11:45 and will be discussed in the tutoral next Tuesday the 16th of May at 13:30 in D404. Please, hand in your solutions in postbox 15 near F411.

1) Given a topological space X and a subset $A \subset X$, prove that the following hold.

a) A point x is a *closure point* of A, i.e. $x \in \bar{A}$, if and only if each neighborhood of x has a nonempty intersection with A.

b) A point x is an *interior point* of A, i.e. $x \in \mathring{A}$, if and only if there exists a neighborhood of x which entirely lies in A.

c) A is closed in X iff $A = \bar{A}$.

d) A is open in X iff $A = \mathring{A}$.

2) a) Let X be a set endowed with the discrete topology. Then the only convergent sequences in X are the ones that are eventually constant, that is, sequences $\{q_i\}_{i\in\mathbb{N}}$ of points in X such that $q_i = q$ for all $i \geq N$ for some $N \in \mathbb{N}$.

b) Let Y be a set endowed with the trivial topology. Then every sequence in Y converges to every point of Y.

3) Let $X_1, \ldots, X_n$ be n topological spaces and let A_i be a subset of X_i for each i. Show that:

a) If A_i is closed in X_i for each i, then $A_1 \times \cdots \times A_n$ is closed in $X_1 \times \cdots \times X_n$ w.r.t. the product topology.

b) $\overline{A_1 \times \cdots \times A_n} = \bar{A}_1 \times \cdots \times \bar{A}_n$.

4) Show the following properties of continuous mappings.

a) Let $f : X \to Y$ be a continuous map between the topological spaces (X, τ_X) and (Y, τ_Y). Let $\mathcal{B}$ be a basis for τ_X and consider the following collection $f(\mathcal{B}) := \{f(B) : B \in \mathcal{B}\}$ of subsets of Y. If f is surjective and open, then $f(\mathcal{B})$ is a basis for τ_Y.

b) Continuous maps preserve the convergence of sequences. That is, if $f : X \to Y$ is a continuous map between two topological spaces (X, τ_X) and (Y, τ_Y) and if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of points in X convergent to a point $x \in X$, then $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $f(x) \in Y$.

5) a) Suppose X is a topological space, and for every $p \in X$ there exists a continuous functional $f : X \to \mathbb{R}$ such that $f^{-1}(\{0\}) = \{p\}$. Then X is Hausdorff.

b) Let X be a set endowed with the trivial topology and Y be any topological space. If Y is Hausdorff, then the only continuous maps $h : X \to Y$ are constant maps.