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TOPOLOGICAL VECTOR SPACES–SS 2017
Exercise Sheet 3

This assignment is due by Friday the 19th of May by 11:45 and will be discussed in the tutoral next Tuesday the 23th of May at 13:30 in D404. Please, hand in your solutions in postbox 15 near F411.

1) Show the following statements using just the definition of t.v.s.

a) Every normed vector space $(X, \|\cdot\|)$ endowed with the topology given by the metric induced by the norm is a t.v.s.. (Hint: use the collection $\{B_r(x_0) := \{x \in X : \|x - x_0\| < r\} : r \in \mathbb{R}^+, x_0 \in X\}$ as a base of the topology).

b) Consider the real vector space $\mathbb{R}$ endowed with the lower limit topology τ generated by the base $B = \{[a, b) : a < b\}$. Show that $(\mathbb{R}, \tau)$ is not a t.v.s..

c) Let us consider on $\mathbb{R}$ the metric:

$$d_h(x, y) := |h(x) - h(y)|, \; \forall x, y \in \mathbb{R}$$

where h is the following function on $\mathbb{R}$:

$$h(x) := \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \\ x & \text{otherwise} \end{cases}$$

Then the metric real vector space $(\mathbb{R}, d_h)$ is not a t.v.s. (Here the field of scalars is also $\mathbb{R}$ but endowed with the usual topology given by the modulus $|\cdot|$).

2) Prove the following statements.

a) The filter $\mathcal{F}(x)$ of neighbourhoods of the point x in a t.v.s. X coincides with the family of the sets $O + x$ for all $O \in \mathcal{F}(o)$, where $\mathcal{F}(o)$ is the filter of neighbourhoods of the origin o (i.e. the neutral element of the vector addition).

b) If B is a balanced subset of a t.v.s. X then so is $\bar{B}$.

c) If B is a balanced subset of a t.v.s. X and $o \in \mathring{B}$ then $\mathring{B}$ is balanced.

3) Prove the following statements.

a) Every t.v.s. has always a base of closed neighbourhoods of the origin.

b) Every t.v.s. has always a base of balanced absorbing neighbourhoods of the origin. In particular, it has always a base of closed balanced absorbing neighbourhoods of the origin.

c) Proper linear subspaces of a t.v.s. are never absorbing. In particular, if M is an open linear subspace of a t.v.s. X then $M = X$.