Universität Konstanz
Fachbereich Mathematik und Statistik
Dr. Maria Infusino
Patick Michalski

TOPOLOGICAL VECTOR SPACES–SS 2017
Exercise Sheet 5

This assignment is due by Friday the 2nd of June by 11:45 and will be discussed in the tutoral next Tuesday the 6th of June at 13:30 in D404. Please, hand in your solutions in postbox 15 near F411.

1) Prove the following statements.

a) If M is a linear dense subspace of a t.v.s X, then the quotient topology on X/M is the trivial topology.

b) If X is a Hausdorff t.v.s., then any linear subspace M of X endowed with the correspondent subspace topology is itself a Hausdorff t.v.s..

c) Give an example of a t.v.s X which is not Hausdorff and of a linear subspace $M \neq \{0\}$ of X such that M endowed with the subspace topology is instead a Hausdorff t.v.s..

2) Let X be a topological space and $\sim$ be any equivalence relation on X. Consider the quotient set $X/\sim$ endowed with the quotient topology. Then:

- the quotient map $\phi : X \to X/\sim$ is continuous.
- the quotient topology on $X/\sim$ is the finest topology on $X/\sim$ s.t. ϕ is continuous.

3) Let M be a linear subspace of a t.v.s. X. Another linear subspace N of X is called an *algebraic supplementary* (or algebraic complement) of M in X if the mapping:

$$S : M \times N \to X;\ (m, n) \mapsto m + n$$

is an algebraic isomorphism between $M \times N$ and X. In this case, X is called *algebraic direct sum* of M and N. N is called a *topological supplementary* (or topological complement) of M in X if S is a topological isomorphism between $M \times N$ and X. In this case, X is called the *topological direct sum* of M and N.

Prove the equivalence of the following two properties:

a) N is a topological supplementary of M in X

b) the restriction to N of the canonical mapping $\phi : X \to X/M$ is a topological isomorphism between N and X/M.

Prove also that M has at least one topological supplementary in X if and only if there is a continuous linear map p of X onto M such that $p \circ p = p$ (then $p(x) = x$ for all $x \in M$).