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Fachbereich Mathematik und Statistik
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TOPOLOGICAL VECTOR SPACES–SS 2017
Exercise Sheet 6

This assignment is due by Tuesday the 13th of June by 11:45 and will be discussed in the tutorial on Tuesday the 20th of June at 13:30 in D404. Please, hand in your solutions in postbox 15 near F411.

1) Let $\mathcal{F}$ be a filter of a topological Hausdorff space X. If $\mathcal{F}$ converges to $x \in X$ and also to $y \in X$, then $x = y$.

2) Let A be a subset of a topological space X. Then $x \in \overline{A}$ if and only if there exists a filter $\mathcal{F}$ of subsets of X such that $A \in \mathcal{F}$ and $\mathcal{F}$ converges to x.

3) Let X be a Hausdorff t.v.s.. Assume that there exists a countable basis $\mathcal{B}$ of neighbourhoods of the origin in X. Prove the following statements:

a) X is complete if and only if X is sequentially complete.

b) Suppose additionally that Y is another t.v.s. (not necessarily with a countable basis). A mapping $f : X \to Y$ (not necessarily linear) is continuous if and only if it is sequentially continuous.

Recall that a mapping f from a topological space X into a topological space Y is said to be *sequentially continuous* if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ convergent to a point $x \in X$ the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $f(x)$ in Y.

4) Let $\mathcal{C}(\mathbb{R})$ be the vector space of real valued functions defined and continuous on the real line, and $\mathcal{C}_c(\mathbb{R})$ the space of functions $f \in \mathcal{C}(\mathbb{R})$ whose support is a compact subset of $\mathbb{R}$. The collection of the sets

$$N_{\varepsilon,n} := \left\{ f \in \mathcal{C}(\mathbb{R}) : \sup_{|t|\leq n} |f(t)| \leq \varepsilon \right\}$$

for all $\varepsilon \in \mathbb{R}^+$ and all $n \in \mathbb{N}$ is a basis of neighbourhoods of the origin for a topology τ on $\mathcal{C}(\mathbb{R})$ which is compatible with the linear structure (given by the pointwise addition and scalar multiplication of functions in $\mathcal{C}(\mathbb{R})$).

Prove that:

a) The t.v.s. $(\mathcal{C}(\mathbb{R}), \tau)$ is a complete Hausdorff space [Hint: use Exercise 1].

b) The linear subspace $\mathcal{C}_c(\mathbb{R})$ is dense in $\mathcal{C}(\mathbb{R})$.

c) $\mathcal{C}(\mathbb{R})$ is topologically isomorphic to the completion of $\mathcal{C}_c(\mathbb{R})$.