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TOPOLOGICAL VECTOR SPACES–SS 2017
Exercise Sheet 7

This assignment is due by Friday the 23rd of June by 11:45 and will be discussed in the tutorial next Tuesday the 27th of June at 13:30 in D404. Please, hand in your solutions in postbox 15 near F411.

1) Let X be a t.v.s. over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with the usual topology given by the modulus, and L a linear functional on X. Assume $L(x) \neq 0$ for some $x \in X$. Then the following are equivalent:

a) L is continuous.

b) The null space $\ker(L)$ is closed in X.

c) $\ker(L)$ is not dense in X.

d) L is bounded in some neighbourhood of the origin in X.

2) Keeping in mind the following definition:

Definition 1. *Let X and Y be two t.v.s. and let A be a subset of X. A mapping $f : A \to Y$ is said to be* uniformly continuous *if for every neighbourhood V of the origin in Y, there exists a neighbourhood U of the origin in X such that for all pairs of elements $x_1, x_2 \in A$ the implication $x_1 - x_2 \in U \Rightarrow f(x_1) - f(x_2) \in V$ holds.*

show that if X and Y are two t.v.s. and A is a subset of X then the following hold.

a) Any uniformly continuous map $f : A \to Y$ is continuous at every point of A.

b) If $f : A \to Y$ is uniformly continuous, then the image under f of a Cauchy filter on A is a Cauchy filter on Y.

c) If A is a linear subspace of X, then every continuous linear map from A to Y is uniformly continuous.

3) Let $\mathcal{C}^1(\mathbb{R})$ be the vector space of real valued functions defined and once continuously differentiable on the real line. For $\varepsilon > 0$ and $n \in \mathbb{N}$, set

$$W_{\varepsilon,n} := \left\{ f \in \mathcal{C}^1(\mathbb{R}) : \sup_{|t| \leq n} \left(|f(t)| + |f'(t)| \right) \leq \varepsilon \right\}.$$

a) Show that the collection of the sets $W_{\varepsilon,n}$ for all $\varepsilon \in \mathbb{R}^+$ and all $n \in \mathbb{N}$ is a basis of neighbourhoods of the origin for a Hausdorff topology on $\mathcal{C}^1(\mathbb{R})$ which is compatible with the linear structure.

b) Consider the t.v.s. $\mathcal{C}(\mathbb{R})$ as defined in Exercise 4 in Sheet 6 and the mapping:

$$\begin{array}{rcl} D : \mathcal{C}^1(\mathbb{R}) & \to & \mathcal{C}(\mathbb{R}) \\ f & \mapsto & D(f) := f'. \end{array}$$

Prove that D is continuous.

4) Let X be a t.v.s. over $\mathbb{R}$ and X^* its algebraic dual. Equip X^* with the topology of pointwise convergence in X. A basis of neighbourhoods of the origin in this topology is provided by the sets

$$W(S, \varepsilon) := \{\ell \in X^* : \sup_{x \in S} |\ell(x)| \leq \varepsilon\},$$

where S ranges over the family of finite subsets of X and $\varepsilon \in \mathbb{R}^+$. Prove that X^* is complete.