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# TOPOLOGICAL VECTOR SPACES–SS 2017
## Exercise Sheet 8

*This assignment is due by Friday the 30th of June by 11:45 and will be discussed in the tutorial next Tuesday the 4th of July at 13:30 in D404. Please, hand in your solutions in postbox 15 near F411.*

**1)** Let $S, T$ be arbitrary subsets of a vector space $X$. Show that the following hold.

a) $conv(S)$ is convex

b) $S \subseteq conv(S)$

c) A set is convex if and only if it is equal to its own convex hull.

d) If $S \subseteq T$ then $conv(S) \subseteq conv(T)$

e) $conv(conv(S)) = conv(S)$.

f) $conv(S + T) = conv(S) + conv(T)$.

g) The convex hull of $S$ is the smallest convex set containing $S$, i.e. $conv(S)$ is the intersection of all convex sets containing $S$.

h) The convex hull of a balanced set is balanced

**2)** Prove the following characterization of locally convex t.v.s (i.e. Theorem 4.1.14 in the lecture notes)

**Theorem 1.** *If $X$ is a l.c. t.v.s. then there exists a basis $\mathcal{B}$ of neighbourhoods of the origin consisting of absorbing absolutely convex subsets s.t.*

*a) $\forall U, V \in \mathcal{B}$, $\exists W \in \mathcal{B}$ s.t. $W \subseteq U \cap V$*

*b) $\forall U \in \mathcal{B}$, $\forall \rho > 0$, $\exists W \in \mathcal{B}$ s.t. $W \subseteq \rho U$*

*Conversely, if $\mathcal{B}$ is a collection of absorbing absolutely convex subsets of a vector space $X$ s.t. a) and b) hold, then there exists a unique topology compatible with the linear structure of $X$ s.t. $\mathcal{B}$ is a basis of neighbourhoods of the origin in $X$ for this topology (which is necessarily locally convex).*

**3)** Let $\mathcal{C}(\mathbb{R})$ be the vector space of all real valued continuous functions on the real line. For any bounded interval $[a, b]$ with $a < b$ and any $p > 0$, we define:

$$q_p(f) := \left( \int_a^b |f(t)|^p dt \right)^{\frac{1}{p}}, \; \forall f \in \mathcal{C}(\mathbb{R}).$$

Show that for any $1 \leq p < \infty$ the function $q_p$ is a seminorm but that if $0 < p < 1$ then $q_p$ is not a seminorm.

**4)** Let $0 < p < 1$ and consider the vector space

$$\ell_p := \{(x_i)_{i \in \mathbb{N}} : \forall i \in \mathbb{N}, x_i \in \mathbb{R} \text{ and } \sum_{i=1}^{\infty} |x_i|^p < \infty\}.$$

For any $x, y \in \ell_p$ define $d(x, y) := |x - y|_p$ where for any $z := (x_i)_{i \in \mathbb{N}} \in \ell_p$ we set $|z|_p := \sum_{i=1}^{\infty} |z_i|^p$. Show that the t.v.s. given by $\ell_p$ endowed with the topology induced by $d$ is not locally convex.