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TOPOLOGICAL VECTOR SPACES–SS 2017
Exercise Sheet 9

This sheet aims to self-assess your progress and to explicitly work out more details of some of the results proposed in the lectures. You do not need to hand in solutions for these exercises but please prepare them by Friday July 7th at 10:00, when they will be discussed in F426.

1) Consider the following theorem (Theorem 4.2.12 in the lecture notes) about the comparison of locally convex topologies

Theorem. *Let $\mathcal{P} = \{p_i\}_{i\in I}$ and $\mathcal{Q} = \{q_j\}_{j\in J}$ be two families of seminorms on the vector space X inducing respectively the topologies $\tau_\mathcal{P}$ and $\tau_\mathcal{Q}$, which both make X into a locally convex t.v.s.. Then $\tau_\mathcal{P}$ is finer than $\tau_\mathcal{Q}$ (i.e. $\tau_\mathcal{Q} \subseteq \tau_\mathcal{P}$) iff*

$$\forall q \in \mathcal{Q}\ \exists n \in \mathbb{N},\ i_1, \dots, i_n \in I,\ C > 0\ \text{ s.t. } Cq(x) \leq \max_{k=1,\dots,n} p_{i_k}(x),\ \forall x \in X. \tag{1}$$

a) Give an alternative proof of this result without using Proposition 4.2.11 in the lecture notes.

b) Show that the theorem still holds if we replace (1) with:

$$\forall q \in \mathcal{Q}\ \exists n \in \mathbb{N},\ i_1, \dots, i_n \in I,\ C > 0 \text{ s.t. } Cq(x) \leq \sum_{k=1}^{n} p_{i_k}(x),\ \forall x \in X.$$

2) Fix some $d \in \mathbb{N}$. For any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ one defines $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. For any $\beta \in \mathbb{N}_0^d$ denote by D^β the partial derivative of order $|\beta|$ where $|\beta| := \sum_{i=1}^d \beta_i$, i.e.

$$D^\beta := \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}} = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_d}}{\partial x_d^{\beta_d}}.$$

(a) Consider the space of infinitely differentiable functions $\mathcal{C}^\infty(\mathbb{R}^d)$. Show that the maps $p_{m,K} : \mathcal{C}^\infty(\mathbb{R}^d) \to \mathbb{R}$ defined by

$$p_{m,K}(f) := \sup_{\beta\in\mathbb{N}_0^d, |\beta|\leq m} \sup_{x\in K} \left|(D^\beta f)(x)\right|$$

for $m \in \mathbb{N}_0$ and $K \subseteq \mathbb{R}^d$ compact are seminorms. Further, show that the topology $\tau_\mathcal{P}$ on $\mathcal{C}^\infty(\mathbb{R}^d)$ induced by this family of seminorms makes $\mathcal{C}^\infty(\mathbb{R}^d)$ into a Hausdorff l.c.t.v.s.

(b) Consider the Schwartz space or space of rapidly decreasing functions on $\mathbb{R}^d$ denoted by $\mathcal{S}(\mathbb{R}^d)$;

$$\mathcal{S}(\mathbb{R}^d) := \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) : \sup_{x\in\mathbb{R}^d} \left| x^\alpha (D^\beta f)(x) \right| < \infty, \forall \alpha, \beta \in \mathbb{N}_0^d \right\}.$$

Show that the maps $q_{\alpha,\beta} : \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ defined by

$$q_{\alpha,\beta}(f) := \sup_{x\in\mathbb{R}^d} \left| x^\alpha (D^\beta f)(x) \right|$$

for $\alpha, \beta \in \mathbb{N}_0^d$ are seminorms. Further, show that the topology τ_Q on $\mathcal{S}(\mathbb{R}^d)$ induced by this family of seminorms makes $\mathcal{S}(\mathbb{R}^d)$ into a Hausdorff locally convex t.v.s.

(c) Endow $\mathcal{S}(\mathbb{R}^d) \subseteq \mathcal{C}^\infty(\mathbb{R}^d)$ with the subspace topology $\tau_{\mathcal{P}}^{\mathcal{S}}$ induced by $\tau_{\mathcal{P}}$. Use Exercise 1) to show that the topology τ_Q is finer than $\tau_{\mathcal{P}}^{\mathcal{S}}$.

3) Let X be a locally convex t.v.s. whose topology is induced by a family of directed family of seminorms $\mathcal{P}$. Show that a basis of neighbourhoods of the origin in X for such a topology is given by:

$$\mathcal{B}_d := \{ r\mathring{U}_p : p \in \mathcal{P}, r > 0 \},$$

where $\mathring{U}_p := \{ x \in X : p(x) < 1 \}$.