2.3 Bounded subsets of special classes of t.v.s.

In this section we are going to study bounded sets in some of the special classes of t.v.s. which we have encountered so far. First of all, let us notice that any ball in a normed space is a bounded set and thus that in normed spaces there exist sets which are at the same time bounded and neighbourhoods of the origin. This property is actually a characteristic of all normable Hausdorff locally convex t.v.s.. Recall that a t.v.s. E is said to be *normable* if its topology can be defined by a norm, i.e. if there exists a norm $\|\cdot\|$ on E such that the collection $\{B_r : r > 0\}$ with $B_r := \{x \in E : \|x\| < r\}$ is a basis of neigbourhoods of the origin in E.

Proposition 2.3.1. *Let E be a Hausdorff locally convex t.v.s.. If there is a neighbourhood of the origin in E which is also bounded, then E is normable.*

Proof. Let U be a bounded neighbourhood of the origin in E. As E is locally convex, by Proposition 4.1.12 in TVS-I, we may always assume that U is open and absolutely convex, i.e. convex and balanced. The boundedness of U implies that for any balanced neigbourhood V of the origin in E there exists $\lambda > 0$ s.t. $U \subseteq \lambda V$. Hence, $U \subseteq nV$ for all $n \in \mathbb{N}$ such that $n \geq \lambda$, i.e. $\frac{1}{n}U \subseteq V$. Then the collection $\left\{\frac{1}{n}U\right\}_{n\in\mathbb{N}}$ is a basis of neighbourhoods of the origin o in E and, since E is a Hausdorff t.v.s., Corollary 2.2.4 in TVS-I guarantees that

$$\bigcap_{n\in\mathbb{N}} \frac{1}{n}U = \{o\}. \tag{2.2}$$

Since E is locally convex and U is an open absolutely convex neighbourhood of the origin, there exists a generating seminorm p on E s.t. $U = \{x \in E : p(x) < 1\}$ (see second part of proof of Theorem 4.2.9 in TVS-I). Then p must be a norm, because $p(x) = 0$ implies $x \in \frac{1}{n}U$ for all $n \in \mathbb{N}$ and so $x = 0$ by (2.2). Hence, E is normable. □

An interesting consequence of this result is the following one.

Corollary 2.3.2. *Let E be a locally convex metrizable space. If E is not normable, then E cannot have a countable basis of bounded sets in E.*

Proof. (Sheet 6, Exercise 1) □

The notion of boundedness can be extended to linear maps between t.v.s..

Definition 2.3.3. *Let E, F be two t.v.s. and f a linear map of E into F. f is said to be* bounded *if for every bounded subset B of E, $f(B)$ is a bounded subset of F.*

We have already showed in Proposition 2.2.9 that any continuous linear map between two t.v.s. is a bounded map. The converse is not true in general but it holds for two special classes of t.v.s.: metrizable t.v.s. and LF-spaces.

Proposition 2.3.4. *Let E be a metrizable t.v.s. and let f be a linear map of E into a t.v.s. F. If f is bounded, then f is continuous.*

Proof. Let $f : E \to F$ be a bounded linear map. Suppose that f is not continuous. Then there exists a neighbourhood V of the origin in F whose preimage $f^{-1}(V)$ is not a neighbourhood of the origin in E. W.l.o.g. we can always assume that V is balanced. As E is metrizable, we can take a countable basis $\{U_n\}_{n\in\mathbb{N}}$ of neighbourhood of the origin in E s.t. $U_n \supseteq U_{n+1}$ for all $n \in \mathbb{N}$. Then for all $m \in \mathbb{N}$ we have $\frac{1}{m}U_m \not\subseteq f^{-1}(V)$ i.e.

$$\forall\, m \in \mathbb{N},\ \exists\, x_m \in \frac{1}{m}U_m \text{ s.t. } f(x_m) \notin V. \tag{2.3}$$

As for all $m \in \mathbb{N}$ we have $mx_m \in U_m$ we get that the sequence $\{mx_m\}_{m\in\mathbb{N}}$ converges to the origin o in E. In fact, for any neighbourhood U of the origin o in E there exists $\bar{n} \in \mathbb{N}$ s.t. $U_{\bar{n}} \subseteq U$. Then for all $n \geq \bar{n}$ we have $x_n \in U_n \subseteq U_{\bar{n}} \subseteq U$, i.e. $\{mx_m\}_{m\in\mathbb{N}}$ converges to o.

Hence, Proposition 2.2.7 implies that $\{mx_m\}_{m\in\mathbb{N}_0}$ is bounded in E and so, since f is bounded, also $\{mf(x_m)\}_{m\in\mathbb{N}_0}$ is bounded in F. This means that there exists $\rho > 0$ s.t. $\{mf(x_m)\}_{m\in\mathbb{N}_0} \subseteq \rho V$. Then for all $n \in \mathbb{N}$ with $n \geq \rho$ we have $f(x_n) \in \frac{\rho}{n}V \subseteq V$ which contradicts (2.3).

□

To show that the previous proposition also hold for LF-spaces, we need to introduce the following characterization of bounded sets in LF-spaces.

Proposition 2.3.5.
Let (E, τ_{ind}) be an LF-space with defining sequence $\{(E_n, \tau_n)\}_{n\in\mathbb{N}}$. A subset B of E is bounded in E if and only if there exists $n \in \mathbb{N}$ s.t. B is contained in E_n and B is bounded in E_n.

To prove this result we will need the following refined version of Lemma 1.3.3.

Lemma 2.3.6. *Let Y be a locally convex space, Y_0 a closed linear subspace of Y equipped with the subspace topology, U a convex neighbourhood of the origin in Y_0, and $x_0 \in Y$ with $x_0 \notin U$. Then there exists a convex neighbourhood V of the origin in Y such that $x_0 \notin V$ and $V \cap Y_0 = U$.*

Proof.
By Lemma 1.3.3 we have that there exists a convex neighbourhood W of the origin in Y such that $W \cap Y_0 = U$. Now we need to distinguish two cases:
-If $x_0 \in Y_0$ then necessarily $x_0 \notin W$ since by assumption $x_0 \notin U$. Hence, we are done by taking $V := W$.
-If $x_0 \notin Y_0$, then let us consider the quotient Y/Y_0 and the canonical map $\phi : Y \to Y/Y_0$. As Y_0 is a closed linear subspace of Y and Y is locally convex, we have that Y/Y_0 is Hausdorff and locally convex. Then, since $\phi(x_0) \neq o$, there exists a convex neighbourhood N of the origin o in Y/Y_0 such that $\phi(x_0) \notin N$. Set $\Omega := \phi^{-1}(N)$. Then Ω is a convex neighbourhood of the origin in Y such that $x_0 \notin \Omega$ and clearly $Y_0 \subseteq \Omega$ (as $\phi(Y_0) = o \in N$). Therefore, if we consider $V := \Omega \cap W$ then we have that: V is a convex neighbourhood of the origin in Y, $V \cap Y_0 = \Omega \cap W \cap Y_0 = W \cap Y_0 = U$ and $x_0 \notin V$ since $x_0 \notin \Omega$. □

Proof. of Proposition 2.3.5

Suppose first that B is contained and bounded in some E_n. Let U be an arbitrary neighbourhood of the origin in E. Then by Proposition 1.3.4 we have that $U_n := U \cap E_n$ is a neighbourhood of the origin in E_n. Since B is bounded in E_n, there is a number $\lambda > 0$ such that $B \subseteq \lambda U_n \subseteq \lambda U$, i.e. B is bounded in E.

Conversely, assume that B is bounded in E. Suppose that B is not contained in any of the E_n's, i.e. $\forall n \in \mathbb{N}, \exists x_n \in B$ s.t. $x_n \notin E_n$. We will show that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is not bounded in E and so a fortiori B cannot be bounded in E.
Since $x_1 \notin E_1$ but $x_1 \in B \subset E$, given an arbitrary convex neighbourhood U_1 of the origin in E_1 we can apply Lemma 2.3.6 and get that there exists U_2' convex neighbourhood of the origin in E s.t. $x_1 \notin U_2'$ and $U_2' \cap E_1 = U_1$. As $\tau_{ind} \restriction E_2 = \tau_2$, we have that $U_2 := U_2' \cap E_2$ is a convex neighbourhood of the origin in E_2 s.t. $x_1 \notin U_2$ and $U_2 \cap E_1 = U_2' \cap E_2 \cap E_1 = U_2' \cap E_1 = U_1$.
Since $x_1 \notin U_2$, we can apply once again Lemma 2.3.6 and proceed as above to get that there exists V_3' convex neighbourhood of the origin in E_3 s.t. $x_1 \notin V_3'$ and $V_3' \cap E_2 = U_2$. Since $x_2 \notin E_2$ we also have $\frac{1}{2}x_2 \notin E_2$ and so $\frac{1}{2}x_2 \notin U_2$. By applying again Lemma 2.3.6 and proceeding as above, we get that there exists V_3 convex neighbourhood of the origin in E_3 s.t. $\frac{1}{2}x_2 \notin V_3$ and $V_3 \cap E_2 = U_2$. Taking $U_3 := V_3 \cap V_3'$ we have that $U_3 \cap E_2 = U_2$ and $x_1, \frac{1}{2}x_2 \notin U_2$.
By induction on n, we get a sequence $\{U_n\}_{n\in\mathbb{N}}$ such that for any $n \in \mathbb{N}$:

- U_n is a convex neighbourhood of the origin in E_n
- $U_n = U_{n+1} \cap E_n$
- $x_1, \frac{1}{2}x_2, \dots, \frac{1}{n}x_n \notin U_{n+1}$.

Note that:
$U_n = U_{n+1} \cap E_n = U_{n+2} \cap E_{n+1} \cap E_n = U_{n+2} \cap E_n = \cdots = U_{n+k} \cap E_n, \quad \forall k \in \mathbb{N}.$

Consider $U := \bigcup_{j=1}^{\infty} U_j$, then for each $n \in \mathbb{N}$ we have

$$U \cap U_n = \left(\bigcup_{j=1}^{n} U_j \cap U_n \right) \cup \left(\bigcup_{j=n+1}^{\infty} U_j \cap U_n \right) = U_n \cup \left(\bigcup_{k=1}^{\infty} U_{n+k} \cap U_n \right) = U_n,$$

i.e. U is a neighbourhood of the origin in (E, τ_{ind}).

Suppose that $\{x_j\}_{j \in \mathbb{N}}$ is bounded in E then it should be swallowed by U. Take a balanced neighbourhood V of the origin in E s.t. $V \subseteq U$. Then there would exists $\lambda > 0$ s.t. $\{x_j\}_{j \in \mathbb{N}} \subseteq \lambda V$ and so $\{x_j\}_{j \in \mathbb{N}} \subseteq nV$ for some $n \in \mathbb{N}$ with $n \geq \lambda$. In particular, we would have $x_n \in nV$ which would imply $\frac{1}{n} x_n \in V \subseteq U$; but this would contradict the third property of the U_j's (i.e. $\frac{1}{n} \notin \bigcup_{j=1}^{\infty} U_{n+j} = \bigcup_{j=n+1}^{\infty} U_j = U$, since $U_j \subseteq U_{j+1}$ for any $j \in \mathbb{N}$). Hence, $\{x_j\}_{j \in \mathbb{N}}$ is not bounded in E and so B cannot be bounded in E. This contradicts our assumption and so proves that $B \subseteq E_n$ for some $n \in \mathbb{N}$.

It remains to show that B is bounded in E_n. Let W_n be a neighbourhood of the origin in E_n. By Proposition 1.3.4, there exists a neighbourhood W of the origin in E such that $W \cap E_n = W_n$. Since B is bounded in E, there exists $\mu > 0$ s.t. $B \subseteq \mu W$ and hence $B = B \cap E_n \subseteq \mu W \cap E_n = \mu(W \cap E_n) = W_n$. □

Corollary 2.3.7. *A bounded linear map from an LF- space into an arbitrary t.v.s. is always continuous.*

Proof. (Sheet 5, Exercise 2) □

Chapter 3

Topologies on the dual space of a t.v.s.

In this chapter we are going to describe a general method to construct a whole class of topologies on the topological dual of a t.v.s. using the notion of polar of a subset. Among these topologies, the so-called polar topologies, there are: the weak topology, the topology of compact convergence and the strong topology.

In this chapter we will denote by:

- E a t.v.s. over the field $\mathbb{K}$ of real or complex numbers.
- E^* the algebraic dual of E, i.e. the vector space of all linear functionals on E.
- E' its topological dual of E, i.e. the vector space of all continuous linear functionals on E.

Moreover, given $x' \in E'$, we denote by $\langle x', x\rangle$ its value at the point x of E, i.e. $\langle x', x\rangle = x'(x)$. The bracket $\langle\cdot,\cdot\rangle$ is often called *pairing* between E and E'.

3.1 The polar of a subset of a t.v.s.

Definition 3.1.1. *Let A be a subset of E. We define the* polar of A *to be the subset A° of E' given by:*

$$A^\circ := \left\{ x' \in E' : \sup_{x\in A} |\langle x', x\rangle| \leq 1 \right\}.$$

Let us list some properties of polars:

a) The polar A° of a subset A of E is a convex balanced subset of E'.
b) If $A \subseteq B \subseteq E$, then $B^\circ \subseteq A^\circ$.
c) $(\rho A)^\circ = (\frac{1}{\rho})A^\circ, \forall\, \rho > 0, \forall\, A \subseteq E$.
d) $(A \cup B)^\circ = A^\circ \cap B^\circ, \forall\, A, B \subseteq E$.
e) If A is a cone in E, then $A^\circ \equiv \{x' \in E' : \langle x', x\rangle = 0, \forall\, x \in A\}$ and A° is a linear subspace of E'. In particular, this property holds when A is a linear

subspace of E and in this case the polar of A is called the *orthogonal of* A, i.e. the set of all continuous linear forms on E which vanish identically in A.

Proof. (Sheet 5, Exercise 3) □

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