Chapter 3

Topologies on the dual space of a t.v.s.

In this chapter we are going to describe a general method to construct a whole class of topologies on the topological dual of a t.v.s. using the notion of polar of a subset. Among these topologies, the so-called polar topologies, there are: the weak topology, the topology of compact convergence and the strong topology.

In this chapter we will denote by:

- E a t.v.s. over the field $\mathbb{K}$ of real or complex numbers.
- E^* the algebraic dual of E, i.e. the vector space of all linear functionals on E.
- E' its topological dual of E, i.e. the vector space of all continuous linear functionals on E.

Moreover, given $x' \in E'$, we denote by $\langle x', x\rangle$ its value at the point x of E, i.e. $\langle x', x\rangle = x'(x)$. The bracket $\langle \cdot, \cdot\rangle$ is often called *pairing* between E and E'.

3.1 The polar of a subset of a t.v.s.

Definition 3.1.1. *Let A be a subset of E. We define the* polar of A *to be the subset A° of E' given by:*

$$A^\circ := \left\{ x' \in E' : \sup_{x\in A} |\langle x', x\rangle| \leq 1 \right\}.$$

Let us list some properties of polars:

a) The polar A° of a subset A of E is a convex balanced subset of E'.

b) If $A \subseteq B \subseteq E$, then $B^\circ \subseteq A^\circ$.

c) $(\rho A)^\circ = (\frac{1}{\rho})A^\circ, \forall\, \rho > 0, \forall\, A \subseteq E$.

d) $(A \cup B)^\circ = A^\circ \cap B^\circ, \forall\, A, B \subseteq E$.

e) If A is a cone in E, then $A^\circ \equiv \{x' \in E' : \langle x', x\rangle = 0, \forall\, x \in A\}$ and A° is a linear subspace of E'. In particular, this property holds when A is a linear subspace of E and in this case the polar of A is called the *orthogonal of* A, i.e. the set of all continuous linear forms on E which vanish identically in A.

Proof. (Sheet 5, Exercise 3) □

Proposition 3.1.2. *Let E be a t.v.s.. If B is a bounded subset of E, then the polar B° of B is an absorbing subset of E'.*

Proof.
Let $x' \in E'$. As B is bounded in E, Corollary 2.2.10 guarantees that any continuous linear functional x' on E is bounded on B, i.e. there exists a constant $M(x') > 0$ such that $\sup_{x\in B} |\langle x', x\rangle| \leq M(x')$. This implies that for any $\lambda \in \mathbb{K}$ with $|\lambda| \leq \frac{1}{M(x')}$ we have $\lambda x' \in B^\circ$, since

$$\sup_{x\in B} |\langle \lambda x', x\rangle| = |\lambda| \sup_{x\in B} |\langle x', x\rangle| \leq \frac{1}{M(x')} \cdot M(x') = 1.$$

□

3.2 Polar topologies on the topological dual of a t.v.s.

We are ready to define an entire class of topologies on the dual E' of E, called *polar topologies*. Consider a family Σ of bounded subsets of E with the following two properties:
(P1) If $A, B \in \Sigma$, then $\exists\, C \in \Sigma$ s.t. $A \cup B \subseteq C$.
(P2) If $A \in \Sigma$ and $\lambda \in \mathbb{K}$, then $\exists\, B \in \Sigma$ s.t. $\lambda A \subseteq B$.
Let us denote by Σ° the family of the polars of the sets belonging to Σ, i.e.

$$\Sigma^\circ := \{A^\circ : \ A \in \Sigma\}.$$

Claim: Σ° is a basis of neighbourhoods of the origin for a locally convex topology on E' compatible with the linear structure.

Proof. of Claim.
By Property a) of polars and by Proposition 3.1.2, all elements of Σ° are convex balanced absorbing susbsets of E'. Also:

1. $\forall\, A^\circ, B^\circ \in \Sigma^\circ$, $\exists\, C^\circ \in \Sigma^\circ$ s.t. $C^\circ \subseteq A^\circ \cap B^\circ$.
   Indeed, if A° and B° in Σ° are respectively the polars of A and B in Σ, then by (P1) there exists $C \in \Sigma$ s.t. $A \cup B \subseteq C$ and so, by properties b) and d) of polars, we get: $C^\circ \subseteq (A \cup B)^\circ = A^\circ \cap B^\circ$.

2. $\forall\, A^\circ \in \Sigma^\circ, \forall\, \rho > 0,\ \exists\, B^\circ \in \Sigma^\circ$ s.t. $B^\circ \subseteq \rho A^\circ$.
   Indeed, if A° in Σ° is the polar of A, then by (P2) there exists $B \in \Sigma$ s.t. $\frac{1}{\rho}A \subseteq B$ and so, by properties b) and c) of polars, we get that $B^\circ \subseteq \left(\frac{1}{\rho}A\right)^\circ = \rho A^\circ$.

By Theorem 4.1.14 in TVS-I, there exists a unique locally convex topology on E' compatible with the linear structure and having Σ° as a basis of neighborhoods of the origin. □

Definition 3.2.1. *Given a family Σ of bounded subsets of a t.v.s. E s.t. (P1) and (P2) hold, we call* $\Sigma-$topology *on E' the locally convex topology defined by taking, as a basis of neighborhoods of the origin in E', the family Σ° of the polars of the subsets that belong to Σ. We denote by E'_Σ the space E' endowed with the Σ-topology.*

It is easy to see from the definition that (Sheet 6, Exercise 1):

- The $\Sigma-$topology on E' is generated by the following family of seminorms:
$$\{p_A :\ A \in \Sigma\}\,,\ \text{where } p_A(x') := \sup_{x\in A} |\langle x', x\rangle|, \forall\, x' \in E'. \tag{3.1}$$
- Define for any $A \in \Sigma$ and $\varepsilon > 0$ the following subset of E':
$$W_\varepsilon(A) := \left\{ x' \in E' : \sup_{x\in A} |\langle x', x\rangle| \leq \varepsilon \right\}.$$
  The family $\mathcal{B} := \{W_\varepsilon(A) :\ A \in \Sigma, \varepsilon > 0\}$ is a basis of neighbourhoods of the origin for the $\Sigma-$topology on E'.

Proposition 3.2.2. *A filter $\mathcal{F}'$ on E' converges to an element $x' \in E'$ in the Σ-topology on E' if and only if $\mathcal{F}'$ converges uniformly to x' on each subset A belonging to Σ, i.e. the following holds:*
$$\forall \varepsilon > 0, \exists M' \in \mathcal{F}' \ s.t.\ \sup_{x\in A} |\langle x', x\rangle - \langle y', x\rangle| \leq \varepsilon,\ \forall\, y' \in M'. \tag{3.2}$$

This proposition explain why the $\Sigma-$topology on E' is often referred as *topology of the uniform converge over the sets of* Σ.

Proof.

Suppose that (3.2) holds and let U be a neighbourhood of the origin in the $\Sigma-$topology on E'. Then there exists $\varepsilon > 0$ and $A \in \Sigma$ s.t. $W_\varepsilon(A) \subseteq U$ and so
$$x' + W_\varepsilon(A) \subseteq x' + U. \tag{3.3}$$

On the other hand, since we have that

$$\begin{aligned} x' + W_\varepsilon(A) &= \left\{ x' + y' \in E' : \sup_{x \in A} |\langle y', x \rangle| \leq \varepsilon \right\} \\ &= \left\{ z' \in E' : \sup_{x \in A} |\langle z' - x', x \rangle| \leq \varepsilon \right\}, \end{aligned} \tag{3.4}$$

the condition (3.2) together with (3.3) gives that

$$\exists M' \in \mathcal{F}' \text{ s.t. } M' \subseteq x' + W_\varepsilon(A) \subseteq x' + U.$$

The latter implies that $x' + U \in \mathcal{F}'$ since $\mathcal{F}'$ is a filter and so the family of all neighbourhoods of x' in the $\Sigma-$topology on E' is contained in $\mathcal{F}'$, i.e. $\mathcal{F}' \to x'$.

Conversely, if $\mathcal{F}' \to x'$, then for any neighbourhood V of x' in the $\Sigma-$topology on E' we have $V \in \mathcal{F}'$. In particular, for all $A \in \Sigma$ and for all $\varepsilon > 0$ we have $x' + W_\varepsilon(A) \in \mathcal{F}'$. Then by taking $M' := x' + W_\varepsilon(A)$ and using (3.4), we easily get (3.2). □

The weak topology on E'

The *weak topology on* E' is the $\Sigma-$topology corresponding to the family Σ of all finite subsets of E and it is usually denoted by $\sigma(E', E)$ (this topology is often also referred with the name of *weak*-topology* or *weak dual topology*). We denote by E'_σ the space E' endowed with the topology $\sigma(E', E)$.

A basis of neighborhoods of $\sigma(E', E)$ is given by the family

$$\mathcal{B}_\sigma := \{W_\varepsilon(x_1, \ldots, x_r) : r \in \mathbb{N}, x_1, \ldots, x_r \in E, \varepsilon > 0\}$$

where

$$W_\varepsilon(x_1, \ldots, x_r) := \left\{ x' \in E' : |\langle x', x_j \rangle| \leq \varepsilon,\ j = 1, \ldots, r \right\}. \tag{3.5}$$

Note that a sequence $\{x'_n\}_{n \in \mathbb{N}}$ of elements in E' converges to the origin in the weak topology if and only if at each point $x \in E$ the sequence of their values $\{\langle x'_n, x \rangle\}_{n \in \mathbb{N}}$ converges to zero in $\mathbb{K}$ (see Sheet 6, Exercise 2). In other words, the weak topology on E' is nothing else but the topology of pointwise convergence in E, when we look at continuous linear functionals on E simply as functions on E.

The topology of compact convergence on E'

The *topology of compact convergence on* E' is the $\Sigma-$topology corresponding to the family Σ of all compact subsets of E and it is usually denoted by $c(E', E)$. We denote by E'_c the space E' endowed with the topology $c(E', E)$.

The strong topology on E'

The *strong topology on* E' is the $\Sigma-$topology corresponding to the family Σ of all bounded subsets of E and it is usually denoted by $b(E', E)$. As a filter in E' converges to the origin in the strong topology if and only if it converges to the origin uniformly on every bounded subset of E (see Proposition 3.2.2), the strong topology on E' is sometimes also referred as *the topology of bounded convergence*. When E' carries the strong topology, it is usually called the *strong dual of* E and denoted by E'_b.

In general we can compare two polar topologies by using the following criterion: *If* Σ_1 *and* Σ_2 *are two families of bounded subsets of a t.v.s.* E *such that (P1) and (P2) hold and* $\Sigma_1 \supseteq \Sigma_2$*, then the* Σ_1*-topology is finer than the* Σ_2*-topology.* In particular, this gives the following comparison relations between the three polar topologies on E' introduced above:

$$\sigma(E', E) \subseteq c(E', E) \subseteq b(E', E).$$

Proposition 3.2.3. *Let* Σ *be a family of bounded subsets of a t.v.s.* E *s.t. (P1) and (P2) hold. If the union of all subsets in* Σ *is dense in* E*, then* E'_Σ *is Hausdorff.*

Proof. Assume that the union of all subsets in Σ is dense in E. As the $\Sigma-$topology is locally convex, to show that E'_Σ is Hausdorff is enough to check that the family of seminorms in (3.1) is separating (see Proposition 4.3.3 in TVS-I). Suppose that $p_A(x') = 0$ for all $A \in \Sigma$, then

$$\sup_{x \in A} |\langle x', x\rangle| = 0, \; \forall A \in \Sigma$$

which gives

$$\langle x', x\rangle = 0, \; \forall x \in \bigcup_{A \in \Sigma} A.$$

As the continuous functional x' is zero on a dense subset of E, it has to be identically zero on the whole E. Hence, the family $\{p_A : A \in \Sigma\}$ is a separating family of seminorms which generates the $\Sigma-$topology on E'. □

Corollary 3.2.4. *The topology of compact convergence, the weak and the strong topologies on* E' *are all Hausdorff.*

Let us consider now for any $x \in E$ the linear functional v_x on E' which associates to each element of the dual E' its *"value at the point* x*"*, i.e.

$$\begin{array}{rccc} v_x : & E' & \to & \mathbb{K} \\ & x' & \mapsto & \langle x', x\rangle. \end{array}$$

Clearly, each $v_x \in (E')^*$ but when can we say that $v_x \in (E'_\Sigma)'$? Can we find conditions on Σ which guarantee the continuity of v_x w.r.t. the $\Sigma-$topology?

Fixed an arbitrary $x \in E$, v_x is continuous on E'_Σ if and only if for any $\varepsilon > 0$, $v_x^{-1}(\bar{B}_\varepsilon(0))$ is a neighbourhood of the origin in E' w.r.t. the $\Sigma-$topology ($\bar{B}_\varepsilon(0)$ denotes the closed ball of radius ε and center 0 in $\mathbb{K}$). This means that

$$\forall\, \varepsilon > 0,\, \exists\, A \in \Sigma: \; A^\circ \subseteq v_x^{-1}(\bar{B}_\varepsilon(0)) = \{x' \in E' : |\langle x', x\rangle| \leq \varepsilon\}$$

i.e.

$$\forall\, \varepsilon > 0,\, \exists\, A \in \Sigma: \left|\langle x', \frac{1}{\varepsilon}x\rangle\right| \leq 1, \forall\, x' \in A^\circ. \tag{3.6}$$

Then it is easy to see that the following holds:

Proposition 3.2.5. *Let Σ be a family of bounded subsets of a t.v.s. E s.t. (P1) and (P2) hold. If Σ covers E then for every $x \in E$ the value at x is a continuous linear functional on E'_Σ, i.e. $v_x \in (E'_\Sigma)'$.*

Proof. If $E \subseteq \bigcup_{A\in\Sigma} A$ then for any $x \in E$ and any $\varepsilon > 0$ we have $\frac{1}{\varepsilon} \in A$ for some $A \in \Sigma$ and so $|\langle x', \frac{1}{\varepsilon}x\rangle| \leq 1$ for all $x' \in A^\circ$. This means that (3.6) is fulfilled, which is equivalent to v_x being continuous w.r.t. the $\Sigma-$topology on E'. □

Remark 3.2.6. *The previous proposition means that, if Σ covers E then the image of E under the canonical map*

$$\begin{array}{rccl} \varphi: & E & \to & (E'_\Sigma)^* \\ & x & \mapsto & v_x. \end{array}$$

is contained in the topological dual of E'_Σ, i.e. $\varphi(E) \subseteq (E'_\Sigma)'$.

Proposition 3.2.5 is useful to get the following characterization of the weak topology on E', which is often taken as a definition for this topology.

Proposition 3.2.7. *The weak topology on E' is the coarsest topology on E' such that, for all $x \in E$, v_x is continuous.*

Proof. (Sheet 6, Exercise 3) □

Bibliography

[1] Y. M. Berezansky. *Selfadjoint Operators in Spaces of Functions of Infinite Many Variables*, volume 63 of *Trans. Amer. Math. Soc.* American Mathematical Society, 1986.

[2] F. Tréves, *Topological Vector Spaces, distributions, and kernels*, Academic Press, 1967