Clearly, each $v_x \in (E')^*$ but when can we say that $v_x \in (E'_\Sigma)'$? Can we find conditions on Σ which guarantee the continuity of v_x w.r.t. the $\Sigma-$topology?

Fixed an arbitrary $x \in E$, v_x is continuous on E'_Σ if and only if for any $\varepsilon > 0$, $v_x^{-1}(\bar{B}_\varepsilon(0))$ is a neighbourhood of the origin in E' w.r.t. the $\Sigma-$topology ($\bar{B}_\varepsilon(0)$ denotes the closed ball of radius ε and center 0 in $\mathbb{K}$). This means that

$$\forall\, \varepsilon > 0,\, \exists\, A \in \Sigma: \; A^\circ \subseteq v_x^{-1}(\bar{B}_\varepsilon(0)) = \{x' \in E' : |\langle x', x\rangle| \leq \varepsilon\}$$

i.e.

$$\forall\, \varepsilon > 0,\, \exists\, A \in \Sigma: \left|\langle x', \frac{1}{\varepsilon}x\rangle\right| \leq 1, \forall\, x' \in A^\circ. \tag{3.6}$$

Then it is easy to see that the following holds:

Proposition 3.2.5. *Let Σ be a family of bounded subsets of a t.v.s. E s.t. (P1) and (P2) hold. If Σ covers E then for every $x \in E$ the value at x is a continuous linear functional on E'_Σ, i.e. $v_x \in (E'_\Sigma)'$.*

Proof. If $E \subseteq \bigcup_{A\in\Sigma} A$ then for any $x \in E$ and any $\varepsilon > 0$ we have $\frac{1}{\varepsilon} \in A$ for some $A \in \Sigma$ and so $|\langle x', \frac{1}{\varepsilon}x\rangle| \leq 1$ for all $x' \in A^\circ$. This means that (3.6) is fulfilled, which is equivalent to v_x being continuous w.r.t. the $\Sigma-$topology on E'. □

The previous proposition is useful to get the following characterization of the weak topology on E', which is often taken as a definition for this topology.

Proposition 3.2.6. *Let E be a t.v.s.. The weak topology on E' is the coarsest topology on E' such that, for all $x \in E$, v_x is continuous.*

Proof. (Sheet 6, Exercise 3) □

Proposition 3.2.5 means that, if Σ covers E then the image of E under the canonical map

$$\begin{array}{rccc} \varphi: & E & \to & (E'_\Sigma)^* \\ & x & \mapsto & v_x. \end{array}$$

is contained in the topological dual of E'_Σ, i.e. $\varphi(E) \subseteq (E'_\Sigma)'$. In general, the canonical map $\varphi : E \to (E'_\Sigma)'$ is neither injective or surjective. However, when we restrict our attention to locally convex Hausdorff t.v.s., the following consequence of Hahn-Banach theorem guarantees the injectivity of the canonical map.

Proposition 3.2.7. *If E is a locally convex Hausdorff t.v.s with $E \neq \{o\}$, then for every $o \neq x_0 \in E$ there exists $x' \in E'$ s.t. $\langle x', x_0\rangle \neq 0$, i.e. $E' \neq \{o\}$.*

Proof.
Let $o \neq x_0 \in E$. Since (E, τ) is a locally convex Hausdorff t.v.s, Proposition 4.3.3 in TVS-I ensures that τ is generated by a separating family $\mathcal{P}$ of seminorms on E and so there exists $p \in \mathcal{P}$ s.t. $p(x_0) \neq 0$. Take $M := span\{x_0\}$ and define the $\ell : M \to \mathbb{K}$ by $\ell(\alpha x_0) := \alpha p(x_0)$ for all $\alpha \in \mathbb{K}$. The functional ℓ is clearly linear and continuous on M. Then by the Hahn-Banach theorem (see Theorem 5.1.1 in TVS-I) we have that there exists a linear functional $x' : E \to \mathbb{K}$ such that $x'(m) = \ell(m)$ for all $m \in M$ and $|x'(x)| \leq p(x)$ for all $x \in E$. Hence, $x' \in E'$ and $\langle x', x_0 \rangle = \ell(x_0) = p(x_0) \neq 0$. □

Corollary 3.2.8. *Let E be a non-trivial locally convex Hausdorff t.v.s and Σ a family of bounded subsets of E s.t. (P1) and (P2) hold and Σ covers E. Then the canonical map $\varphi : E \to (E'_\Sigma)'$ is injective.*

Proof. Let $o \neq x_0 \in E$. By Proposition 3.2.7, we know that there exists $x' \in E'$ s.t. $v_x(x') \neq 0$ which proves that v_x is not identically zero on E' and so that $Ker(\varphi) = \{o\}$. Hence, φ is injective. □

In the particular case of the weak topology on E' the canonical map $\varphi : E \to (E'_\sigma)'$ is also surjective, and so E can be regarded as the dual of its weak dual E'_σ. To show this result we will need to use the following consequence of Hahn-Banach theorem:

Lemma 3.2.9. *Let Y be a closed linear subspace of a locally convex t.v.s. X. If $Y \neq X$, then there exists $f \in X'$ s.t. f is not identically zero on X but identically vanishes on Y.*

Proposition 3.2.10. *Let E be a locally convex Hausdorff t.v.s. Then the canonical map $\varphi : E \to (E'_\sigma)'$ is an isomorphism.*

Proof. Let $L \in (E'_\sigma)'$. By the definition of $\sigma(E', E)$ and Proposition 4.6.1 in TVS-I, we have that there exist $F \subset E$ with $|F| < \infty$ and $C > 0$ s.t.

$$|L(x')| \leq C p_F(x') = C \sup_{x \in F} |\langle x', x \rangle|. \tag{3.7}$$

Take $M := span(F)$ and $d := dim(M)$. Consider an algebraic basis $\mathcal{B} := \{e_1, \ldots, e_d\}$ of M and for each $j \in \{1, \ldots, d\}$ apply Lemma 3.2.9 to $Y := \overline{span\{\mathcal{B} \setminus \{e_j\}\}}$ and $X := M$. Then for each $j \in \{1, \ldots, d\}$ there exists $f_j : M \to \mathbb{K}$ linear and continuous such that $\langle f_j, e_k \rangle = 0$ if $k \neq j$ and $\langle f_j, e_j \rangle \neq 0$. W.l.o.g. we can assume $\langle f_j, e_j \rangle = 1$. By applying Hanh-Banach theorem (see Theorem 5.1.1 in TVS-I), we get that for each $j \in \{1, \ldots, d\}$ there exists e'_j :

$E \to \mathbb{K}$ linear and continuous such that $e'_j \restriction_M = f_j$, in particular $\langle e'_j, e_k \rangle = 0$ for $k \neq j$ and $\langle e'_j, e_j \rangle = 1$.

Let $M' := span\{e'_1, \ldots, e'_d\} \subset E'$, $x_L := \sum_{j=1}^{d} L(e'_j)e_j \in M$ and for any $x' \in E'$ define $p(x') := \sum_{j=1}^{d} \langle x', e_j \rangle e'_j \in M'$. Then for any $x' \in E'$ we get that:

$$\langle x', x_L \rangle = \sum_{j=1}^{d} L(e'_j)\langle x', e_j \rangle = L(p(x')) \tag{3.8}$$

and also

$$\langle x' - p(x'), e_k \rangle = \langle x', e_k \rangle - \sum_{j=1}^{d} \langle x', e_j \rangle \langle e'_j, e_k \rangle = \langle x', e_k \rangle - \langle x', e_k \rangle \langle e_k, e_k \rangle = 0$$

which gives

$$\langle x' - p(x'), m \rangle = 0, \forall m \in M. \tag{3.9}$$

Then for all $x' \in E'$ we have:

$$|L(x' - p(x'))| \overset{(3.7)}{\leq} C \sup_{x \in F} |\langle x' - p(x'), x \rangle| \overset{(3.9)}{=} 0$$

which give that $L(x') = L(p(x')) \overset{(3.8)}{=} \langle x', x_L \rangle = v_{x_L}(x')$. Hence, we have proved that for every $L \in (E'_\sigma)'$ there exists $x_L \in E$ s.t. $\varphi(x_L) \equiv v_{x_L} \equiv L$, i.e. $\varphi : E \to (E'_\sigma)'$ is surjective. Then we are done because the injectivity of $\varphi : E \to (E'_\sigma)'$ follows by applying Corollary 3.2.8 to this special case. □

Remark 3.2.11. *The previous result suggests that it is indeed more convenient to restrict our attention to locally convex Hausdorff t.v.s. when dealing with weak duals. Moreover, as showed in Proposition 3.2.7, considering locally convex Hausdorff t.v.s has the advantage of avoiding the pathological situation in which the topological dual of a non-trivial t.v.s. is reduced to the only zero functional (for an example of a t.v.s. on which there are no continuous linear functional than the trivial one, see Exercise 4 in Sheet 6).*

3.3 The polar of a neighbourhood in a locally convex t.v.s.

Let us come back now to the study of the weak topology and prove one of the milestones of the t.v.s. theory: the *Banach-Alaoglu-Bourbaki theorem*. To prove this important result we need to look for a moment at the algebraic dual E^* of a t.v.s. E. In analogy to what we did in the previous section, we can define *the weak topology on the algebraic dual* E^* (which we will denote by $\sigma(E^*, E)$) as the coarsest topology such that for any $x \in E$ the linear functional w_x is continuous, where

$$\begin{array}{rccl} w_x: & E^* & \to & \mathbb{K} \\ & x^* & \mapsto & \langle x^*, x\rangle := x^*(x). \end{array} \tag{3.10}$$

(Note that $w_x \restriction E' = v_x$). Equivalently, the weak topology on the algebraic dual E^* is the locally convex topology on E^* generated by the family $\{q_F : F \subseteq E, |F| < \infty\}$ of seminorms $q_F(x^*) := \sup_{x\in F} |\langle x^*, x\rangle|$ on E^*. It is then easy to see that $\sigma(E', E) = \sigma(E^*, E) \restriction E'$.

An interesting property of the weak topology on the algebraic dual of a t.v.s. is the following one:

Proposition 3.3.1. *If E is a t.v.s. over $\mathbb{K}$, then its algebraic dual E^* endowed with the weak topology $\sigma(E^*, E)$ is topologically isomorphic to the product of $dim(E)$ copies of the field $\mathbb{K}$ endowed with the product topology.*

Proof.
Let $\{e_i\}_{i\in I}$ be an algebraic basis of E, i.e. $\forall\, x \in E$, $\exists\, \{x_i\}_{i\in I} \in \mathbb{K}^{dim(E)}$ s.t. $x = \sum_{i\in I} x_i e_i$. For any linear functions $L : E \to \mathbb{K}$ and any $x \in E$ we then have $L(x) = \sum_{i\in I} x_i L(e_i)$. Hence, L is completely determined by the sequence $\{L(e_i)\}_{i\in I} \in \mathbb{K}^{dim(E)}$. Conversely, every element $u := \{u_i\}_{i\in I} \in \mathbb{K}^{dim(E)}$ uniquely defines the linear functional L_u on E via $L_u(e_i) := u_i$ for all $i \in I$. This completes the proof that E^* is algebraically isomorphic to $\mathbb{K}^{dim(E)}$. Moreover, the collection $\{W_\varepsilon(e_{i_1}, \dots, e_{i_r}) : \varepsilon > 0,\, r \in \mathbb{N},\, i_1, \dots, i_r \in I\}$, where

$$W_\varepsilon(e_{i_1}, \dots, e_{i_r}) := \{x^* \in E^* : |\langle x^*, e_{i_j}\rangle| \leq \varepsilon, \text{ for } j = 1, \dots, r\},$$

is a basis of neighbourhoods of the origin in $(E^*, \sigma(E^*, E))$. Via the isomorphism described above, we have that for any $\varepsilon > 0$, $r \in \mathbb{N}$, and $i_1, \dots, i_r \in I$:

$$\begin{aligned} W_\varepsilon(e_{i_1}, \dots, e_{i_r}) &\approx \left\{\{u_i\}_{i\in I} \in \mathbb{K}^{dim(E)} : |u_{i_j}| \leq \varepsilon, \text{ for } j = 1, \dots, r\right\} \\ &= \prod_{j=1}^{r} \bar{B}_\varepsilon(0) \times \prod_{I\setminus\{i_1,\dots,i_r\}} \mathbb{K} \end{aligned}$$

and so $W_\varepsilon(e_{i_1}, \dots, e_{i_r})$ is a neighbourhood of the product topology τ_{prod} on $\mathbb{K}^{dim(E)}$ (recall that we always consider the euclidean topology on $\mathbb{K}$). Therefore, $(E^*, \sigma(E^*, E))$ is topological isomorphic to $\left(\mathbb{K}^{dim(E)}, \tau_{prod}\right)$. □

Let us now focus our attention on the polar of a neighbourhood U of the origin in a non-trivial locally convex Hausdorff t.v.s. E. We are considering here only non-trivial locally convex Hausdorff t.v.s. in order to be sure to have non-trivial continuous linear functionals (see Remark 3.2.11) and so to make a meaningful analysis on the topological dual.

First of all let us observe that:

$$\{x^* \in E^* : \sup_{x \in U} |\langle x^*, x \rangle| \leq 1\} \equiv U^\circ := \{x' \in E' : \sup_{x \in U} |\langle x', x \rangle| \leq 1\}. \quad (3.11)$$

Indeed, since $E' \subseteq E^*$, we clearly have $U^\circ \subseteq \{x^* \in E^* : \sup_{x \in U} |\langle x^*, x \rangle| \leq 1\}$. Moreover, any linear functional $x^* \in E^*$ s.t. $\sup_{x \in A} |\langle x^*, x \rangle| \leq 1$ is continuous on E and it is therefore an element of E'.

It is then quite straightforward to show that:

Proposition 3.3.2. *The polar of a neighbourhood U of the origin in E is closed w.r.t. $\sigma(E^*, E)$.*

Proof. By (3.11) and (3.10), it is clear that $U^\circ = \bigcap_{x \in A} w_x^{-1}([-1, 1])$. On the other hand, by definition of $\sigma(E^*, E)$ we have that w_x is continuous on $(E^*, \sigma(E^*, E))$ for all $x \in E$ and so each $w_x^{-1}([-1, 1])$ is closed in $(E^*, \sigma(E^*, E))$. Hence, U° is closed in $(E^*, \sigma(E^*, E))$ as the intersection of closed subsets of $(E^*, \sigma(E^*, E))$. □

We are ready now to prove the famous Banach-Alaoglu-Bourbaki Theorem

Theorem 3.3.3 (Banach-Alaoglu-Bourbaki Theorem)**.**
The polar of a neighbourhood U of the origin in a locally convex Hausdorff t.v.s. $E \neq \{o\}$ is compact in E'_σ.

Proof.
Since U is a neighbourhood of the origin in E, U is absorbing in E, i.e. $\forall\, x \in E$, $\exists\, M_x > 0$ s.t. $M_x x \in U$. Hence, for all $x \in E$ and all $x' \in U^\circ$ we have $|\langle x', M_x x \rangle| \leq 1$, which is equivalent to:

$$\forall\, x \in E, \forall\, x' \in U^\circ, \ |\langle x', x \rangle| \leq \frac{1}{M_x}. \quad (3.12)$$

For any $x \in E$, the subset

$$D_x := \left\{ \alpha \in \mathbb{K} : \ |\alpha| \leq \frac{1}{M_x} \right\}$$

is compact in $\mathbb{K}$ w.r.t. to the euclidean topology and so by Tychnoff's theorem[1] the subset $P := \prod_{x\in E} D_x$ is compact in $\left(\mathbb{K}^{dim(E)}, \tau_{prod}\right)$.

Using the isomorphism introduced in Proposition 3.3.1 and (3.11), we get that

$$U^\circ \approx \{(\langle x^*, x\rangle)_{x\in E} : x^* \in U^\circ\}$$

and so by (3.12) we have that $U^\circ \subset P$. Since Corollary 3.3.2 and Proposition 3.3.1 ensure that U° is closed in $\left(\mathbb{K}^{dim(E)}, \tau_{prod}\right)$, we get that U° is a closed subset of P. Hence, by Proposition 2.1.4–1, U° is compact $\left(\mathbb{K}^{dim(E)}, \tau_{prod}\right)$ and so in $(E^*, \sigma(E^*, E))$. As $U^\circ = E' \cap U^\circ$ we easily see that U° is compact in $(E', \sigma(E', E))$. □

[1]**Tychnoff's theorem**: The product of an arbitrary family of compact spaces endowed with the product topology is also compact.