is compact in $\mathbb{K}$ w.r.t. to the euclidean topology and so by Tychnoff's theorem[1] the subset $P := \prod_{x\in E} D_x$ is compact in $\left(\mathbb{K}^{dim(E)}, \tau_{prod}\right)$.

Using the isomorphism introduced in Proposition 3.3.1 and (3.11), we get that

$$U^\circ \approx \{(\langle x^*, x\rangle)_{x\in E} : x^* \in U^\circ\}$$

and so by (3.12) we have that $U^\circ \subset P$. Since Corollary 3.3.2 and Proposition 3.3.1 ensure that U° is closed in $\left(\mathbb{K}^{dim(E)}, \tau_{prod}\right)$, we get that U° is a closed subset of P. Hence, by Proposition 2.1.4–1, U° is compact $\left(\mathbb{K}^{dim(E)}, \tau_{prod}\right)$ and so in $(E^*, \sigma(E^*, E))$. As $U^\circ = E' \cap U^\circ$ we easily see that U° is compact in $(E', \sigma(E', E))$. □

We briefly introduce now a nice consequence of Banach-Alaoglu-Bourbaki theorem. Let us start by introducing a norm on the topological dual space E' of a seminormed space (E, ρ):

$$\rho'(x') := \sup_{x\in E:\rho(x)\leq 1} |\langle x', x\rangle|.$$

ρ' is usually called the *operator norm* on E'.

Corollary 3.3.4. *Let (E, ρ) be a normed space. The closed unit ball in E' w.r.t. the operator norm ρ' is compact in E'_σ.*

Proof. First of all, let us note that a normed space it is indeed a locally convex Hausdorff t.v.s.. Then, by applying Banach-Alaoglu-Borubaki theorem to the closed unit ball $\bar{B}_1(o)$ in (E, ρ), we get that $\left(\bar{B}_1(o)\right)^\circ$ is compact in E'_σ. The conclusion then easily follow by the observation that $\left(\bar{B}_1(o)\right)^\circ$ actually coincides with the closed unit ball in (E', ρ'):

$$\begin{aligned}
\left(\bar{B}_1(o)\right)^\circ &= \{x' \in E' : \sup_{x\in \bar{B}_1(o)} |\langle x', x\rangle| \leq 1\} \\
&= \{x' \in E' : \sup_{x\in E', \rho(x)\leq 1} |\langle x', x\rangle| \leq 1\} \\
&= \{x' \in E' : \rho'(x') \leq 1\}.
\end{aligned}$$

□

[1]**Tychnoff's theorem**: The product of an arbitrary family of compact spaces endowed with the product topology is also compact.

Chapter 4

Tensor products of t.v.s.

4.1 Tensor product of vector spaces

As usual, we consider only vector spaces over the field $\mathbb{K}$ of real numbers or of complex numbers.

Definition 4.1.1.
*Let E, F, M be three vector spaces over $\mathbb{K}$ and $\phi : E \times F \to M$ be a bilinear map. E and F are said to be ϕ-*linearly disjoint* if:*

(LD) *For any $r \in \mathbb{N}$, any $\{x_1, \ldots, x_r\}$ finite subset of E and any $\{y_1, \ldots, y_r\}$ finite subset of F s.t. $\sum_{i=1}^{r} \phi(x_i, y_j) = 0$, we have that both the following conditions hold:*

- *if $x_1, \ldots, x_r$ are linearly independent in E, then $y_1 = \cdots = y_r = 0$*
- *if $y_1, \ldots, y_r$ are linearly independent in F, then $x_1 = \cdots = x_r = 0$*

Recall that, given three vector spaces over $\mathbb{K}$, a map $\phi : E \times F \to M$ is said to be *bilinear* if:

$$\forall\, x_0 \in E, \quad \phi_{x_0} : \begin{array}{ccc} F & \to & M \\ y & \to & \phi(x_0, y) \end{array} \qquad \text{is linear}$$

and

$$\forall\, y_0 \in F, \quad \phi_{y_0} : \begin{array}{ccc} E & \to & M \\ x & \to & \phi(x, y_0) \end{array} \qquad \text{is linear.}$$

Let us give a useful characterization of $\phi-$linear disjointness.

Proposition 4.1.2. *Let E, F, M be three vector spaces, and $\phi : E \times F \to M$ be a bilinear map. Then E and F are $\phi-$linearly disjoint if and only if:*

(LD') *For any $r, s \in \mathbb{N}$, $x_1, \ldots, x_r$ linearly independent in E and $y_1, \ldots, y_s$ linearly independent in F, the set $\{\phi(x_i, y_j) : i = 1, \ldots, r,\ j = 1, \ldots, s\}$ consists of linearly independent vectors in M.*

Proof.

($\Rightarrow$) Let $x_1, \dots, x_r$ be linearly independent in E and $y_1, \dots, y_s$ be linearly independent in F. Suppose that $\sum_{i=1}^{r}\sum_{j=1}^{s}\lambda_{ij}\phi(x_i, y_j) = 0$ for some $\lambda_{ij} \in \mathbb{K}$. Then, using the bilinearity of ϕ and setting $z_i := \sum_{j=1}^{s}\lambda_{ij}y_j$, we easily get $\sum_{i=1}^{r}\phi(x_i, z_i) = 0$. As the x_i's are linearly independent in E, we derive from (LD) that all z_i's have to be zero. This means that for each $i \in \{1, \dots, r\}$ we have $\sum_{j=1}^{s}\lambda_{ij}y_j = 0$, which implies by the linearly independence of the y_j's that $\lambda_{ij} = 0$ for all $i \in \{1, \dots, r\}$ and all $j \in \{1, \dots, s\}$.

($\Leftarrow$) Let $r \in \mathbb{N}$, $\{x_1, \dots, x_r\} \subseteq E$ and $\{y_1, \dots, y_r\} \subseteq F$ be such that $\sum_{i=1}^{r}\phi(x_i, y_i) = 0$. Suppose that the x_is are linearly independent and let $\{z_1, \dots, z_s\}$ be a basis of $span\{y_1, \dots, y_r\}$. Then for each $i \in \{1, \dots, r\}$ there exist $\lambda_{ij} \in \mathbb{K}$ s.t. $y_i = \sum_{j=1}^{s}\lambda_{ij}z_j$ and so by the bilinearity of ϕ we get:

$$0 = \sum_{i=1}^{r}\phi(x_i, y_j) = \sum_{i=1}^{r}\sum_{j=1}^{s}\lambda_{ij}\phi(x_i, z_j). \tag{4.1}$$

By applying (LD') to the x_i's and $z_j's$, we get that all $\phi(x_i, z_j)$'s are linearly independent. Therefore, (4.1) gives that $\lambda_{ij} = 0$ for all $i \in \{1, \dots, r\}$ and all $j \in \{1, \dots, s\}$ and so $y_i = 0$ for all $i \in \{1, \dots, r\}$. Exchanging the roles of the x_i's and the y_i's we get that (LD) holds. $\square$

Definition 4.1.3. *A tensor product of two vector spaces E and F over $\mathbb{K}$ is a pair (M, ϕ) consisting of a vector space M over $\mathbb{K}$ and of a bilinear map $\phi : E \times F \to M$ (canonical map) s.t. the following conditions are satisfied:*
(TP1) *The image of $E \times F$ spans the whole space M.*
(TP2) *E and F are $\phi-$linearly disjoint.*

We now show that the tensor product of any two vector spaces always exists, satisfies the "universal property" and it is unique up to isomorphisms. For this reason, the tensor product of E and F is usually denoted by $E \otimes F$ and the canonical map by $(x, y) \mapsto x \otimes y$.

Theorem 4.1.4. *Let E, F be two vector spaces over $\mathbb{K}$.*
(a) There exists a tensor product of E and F.
(b) Let (M, ϕ) be a tensor product of E and F. Let G be any vector space over $\mathbb{K}$, and b any bilinear mapping of $E \times F$ into G. There exists a unique linear map $\tilde{b} : M \to G$ such that the following diagram is commutative.

$$\begin{tikzcd} E \times F \arrow[r, "b"] \arrow[d, "\phi"] & G \\ M \arrow[ur, "\tilde{b}"'] & \end{tikzcd}$$

(c) *If (M_1, ϕ_1) and (M_2, ϕ_2) are two tensor products of E and F, then there is a bijective linear map u such that the following diagram is commutative.*

$$\begin{tikzcd} E \times F \arrow[r, "\phi_2"] \arrow[d, "\phi_1"] & M_2 \\ M_1 \arrow[ur, "u"'] & \end{tikzcd}$$

Proof.

(a) Let $\mathcal{H}$ be the vector space of all functions from $E \times F$ into $\mathbb{K}$ which vanish outside a finite set ($\mathcal{H}$ is often called the free space of $E \times F$). For any $(x, y) \in E \times F$, let us define the function $e_{(x,y)} : E \times F \to \mathbb{K}$ as follows:

$$e_{(x,y)}(z, w) := \begin{cases} 1 & \text{if}(z, w) = (x, y) \\ 0 & \text{otherwise} \end{cases} .$$

Then $\mathcal{B}_{\mathcal{H}} := \{e_{(x,y)} : (x, y) \in E \times F\}$ forms a basis of $\mathcal{H}$, i.e.

$$\forall\, h \in \mathcal{H}, \exists! \lambda_{xy} \in \mathbb{K} : h = \sum_{x\in E} \sum_{y\in F} \lambda_{xy} e_{(x,y)}.$$

Let us consider now the following linear subspace of $\mathcal{H}$:

$$N := span \left\{ e_{\left(\sum\limits_{i=1}^{n} a_i x_i, \sum\limits_{j=1}^{m} b_j y_j\right)} - \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j e_{(x_i, y_j)} \,:\, n, m \in \mathbb{N}, a_i, b_j \in \mathbb{K}, (x_i, y_j) \in E \times F \right\}.$$

We then denote by M the quotient vector space $\mathcal{H}/N$, by π the quotient map from $\mathcal{H}$ onto M and by

$$\begin{array}{rccl} \phi : & E \times F & \to & M \\ & (x, y) & \to & \phi(x, y) := \pi\left(e_{(x,y)}\right). \end{array}$$

It is easy to see that the map ϕ is bilinear. Let us just show the linearity in the first variable as the proof in the second variable is just symmetric. Fixed $y \in F$, for any $a, b \in \mathbb{K}$ and any $x_1, x_2 \in E$, we get that:

$$\begin{aligned} \phi(ax_1 + bx_2, y) - a\phi(x_1, y) - b\phi(x_2, y) &= \pi\left(e_{(ax_1+bx_2,y)}\right) - a\pi\left(e_{(x_1,y)}\right) - b\pi\left(e_{x_2,y)}\right) \\ &= \pi\left(e_{(ax_1+bx_2,y)} - ae_{(x_1,y)} - be_{(x_2,y)}\right) \\ &= 0, \end{aligned}$$

where the last equality holds since $e_{(ax_1+bx_2,y)} - ae_{(x_1,y)} - be_{(x_2,y)} \in N$.

We aim to show that (M, ϕ) is a tensor product of E and F. It is clear from the definition of ϕ that

$$span(\phi(E \times F)) = span(\pi(\mathcal{B}_{\mathcal{H}})) = \pi(\mathcal{H}) = M,$$

i.e. (TP1) holds. It remains to prove that E and F are $\phi-$linearly disjoint. Let $r \in \mathbb{N}$, $\{x_1, \dots, x_r\} \subseteq E$ and $\{y_1, \dots, y_r\} \subseteq F$ be such that $\sum_{i=1}^r \phi(x_i, y_i) = 0$. Suppose that the y_i's are linearly independent. For any $\varphi \in E^*$, let us define the linear mapping $A_\phi : \mathcal{H} \to F$ by setting $A_\varphi(e_{(x,y)}) := \varphi(x)y$. Then it is easy to check that A_φ vanishes on N, so it induces a map $\tilde{A}_\varphi : M \to F$ s.t. $\tilde{A}_\varphi(\pi(f)) = A(f)$, $\forall f \in \mathcal{H}$. Hence, since $\sum_{i=1}^r \phi(x_i, y_i) = 0$ can be rewritten as $\pi\left(\sum_{i=1}^r e_{(x_i,y_i)}\right) = 0$, we get that

$$0 = \tilde{A}_\varphi\left(\pi\left(\sum_{i=1}^r e_{(x_i,y_i)}\right)\right) = A_\varphi\left(\sum_{i=1}^r e_{(x_i,y_i)}\right) = \sum_{i=1}^r A_\varphi(e_{(x_i,y_i)}) = \sum_{i=1}^r \varphi(x_i)y_i.$$

This together with the linear independence of the y_i's implies $\varphi(x_i) = 0$ for all $i \in \{1, \dots, r\}$. Since the latter holds for all $\varphi \in E^*$, we have that $x_i = 0$ for all $i \in \{1, \dots, r\}$. Exchanging the roles of the x_i's and the y_i's we get that (LD) holds, and so does (TP2) .

(b) Let (M, ϕ) be a tensor product of E and F, G a vector space and $b : E \times F \to G$ a bilinear map. Consider $\{x_\alpha\}_{\alpha \in A}$ and $\{y_\beta\}_{\beta \in B}$ bases of E and F, respectively. We know that $\{\phi(x_\alpha, y_\beta) : \alpha \in A, \beta \in B\}$ forms a basis of M, as $span(\phi(E \times F)) = M$ and, by Proposition 4.1.2, (LD') holds so the $\phi(x_\alpha, y_\beta)$'s for all $\alpha \in A$ and all $\beta \in B$ are linearly independent. The linear mapping $\tilde{b}$ will therefore be the unique linear map of M into G such that

$$\forall\, \alpha \in A, \forall\, \beta \in B, \ \tilde{b}(\phi(x_\alpha, y_\beta)) = b(x_\alpha, y_\beta).$$

Hence, the diagram in (b) commutes.

(c) Let (M_1, ϕ_1) and (M_2, ϕ_2) be two tensor products of E and F. Then using twice the universal property (b) we get that there exist unique linear maps $u : M_1 \to M_2$ and $v : M_2 \to M_1$ such that the following diagrams both commute:

$$\begin{tikzcd} E \times F \arrow[r, "\phi_2"] \arrow[d, "\phi_1"'] & M_2 \\ M_1 \arrow[ur, "u"'] & \end{tikzcd} \qquad \begin{tikzcd} E \times F \arrow[r, "\phi_1"] \arrow[d, "\phi_2"'] & M_1 \\ M_2 \arrow[ur, "v"'] & \end{tikzcd}$$

Then combining $u \circ \phi_1 = \phi_2$ with $v \circ \phi_2 = \phi_1$, we get that u and v are one the inverse of the other. Hence, there is an algebraic isomorphism between M_1 and M_2. $\square$

It is now natural to introduce the concept of tensor product of linear maps.

Proposition 4.1.5. *Let E, F, E_1, F_1 be four vector spaces over $\mathbb{K}$, and let $u : E \to E_1$ and $v : F \to F_1$ be linear mappings. There is a unique linear map of $E \otimes F$ into $E_1 \otimes F_1$, called the* tensor product of u and v *and denoted by $u \otimes v$, such that*

$$(u \otimes v)(x \otimes y) = u(x) \otimes v(y), \quad \forall\, x \in E, \forall\, y \in F.$$

Proof.
Let us define the mapping

$$\begin{array}{rlcl} b: & E \times F & \to & E_1 \otimes F_1 \\ & (x, y) & \mapsto & b(x, y) := u(x) \otimes v(y), \end{array}$$

which is clearly bilinear because of the linearity of u and v and the bilinearity of the canonical map of the tensor product $E_1 \otimes F_1$. Then by the universal property there is a unique linear map $\tilde{b} : E \otimes F \to E_1 \otimes F_1$ s.t. the following diagram commutes:

$$\begin{tikzcd} E \times F \arrow[r, "b"] \arrow[d, "\otimes"] & E_1 \otimes F_1 \\ E \otimes F \arrow[ur, "\tilde{b}"'] & \end{tikzcd}$$

i.e. $\tilde{b}(x \otimes y) = b(x, y), \forall\,(x, y) \in E \times F$. Hence, using the definition of b, we get that $\tilde{b} \equiv u \otimes v$. $\square$

Examples 4.1.6.

1. *Let $n, m \in \mathbb{N}$, $E = \mathbb{K}^n$ and $F = \mathbb{K}^m$. Then $E \otimes F = \mathbb{K}^{nm}$ is a tensor product of E and F whose canonical bilinear map ϕ is given by:*

$$\begin{array}{rlcl} \phi: & E \times F & \to & \mathbb{K}^{nm} \\ & \Big((x_i)_{i=1}^n, (y_j)_{j=1}^m\Big) & \mapsto & (x_i y_j)_{1 \leq i \leq n, 1 \leq j \leq m}. \end{array}$$

2. *Let X and Y be two sets. For any functions $f : X \to \mathbb{K}$ and $g : Y \to \mathbb{K}$, we define:*

$$\begin{array}{rccc} f \otimes g: & X \times Y & \to & \mathbb{K} \\ & (x, y) & \mapsto & f(x)g(y). \end{array}$$

Let E (resp. F) be the linear space of all functions from X (resp. Y) to $\mathbb{K}$ endowed with the usual addition and multiplication by scalars. We denote by $E \otimes F$ the linear subspace of the space of all functions from $X \times Y$ to $\mathbb{K}$ spanned by the elements of the form $f \otimes g$ for all $f \in E$ and $g \in F$. Then $E \otimes F$ is actually a tensor product of E and F.

Given X and Y open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, we can use the definitions in Example 2 above to construct the tensors $\mathcal{C}^k(X) \otimes \mathcal{C}^l(Y)$ for any $1 \leq k, l \leq \infty$. The approximation results in Section 1.5 imply easily the following:

Theorem 4.1.7. *Let X and Y open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Then $\mathcal{C}_c^\infty(X) \otimes \mathcal{C}_c^\infty(Y)$ is sequentially dense in $\mathcal{C}_c^\infty(X \times Y)$.*