2. *Let X and Y be two sets. For any functions $f : X \to \mathbb{K}$ and $g : Y \to \mathbb{K}$, we define:*

$$\begin{array}{rccl} f \otimes g : & X \times Y & \to & \mathbb{K} \\ & (x, y) & \mapsto & f(x)g(y). \end{array}$$

Let E (resp. F) be the linear space of all functions from X (resp. Y) to $\mathbb{K}$ endowed with the usual addition and multiplication by scalars. We denote by $E \otimes F$ the linear subspace of the space of all functions from $X \times Y$ to $\mathbb{K}$ spanned by the elements of the form $f \otimes g$ for all $f \in E$ and $g \in F$. Then $E \otimes F$ is actually a tensor product of E and F (see Sheet 7, Exercise 1).

Given X and Y open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, we can use the definitions in Example 2 above to construct the tensors $\mathcal{C}^k(X) \otimes \mathcal{C}^l(Y)$ for any $1 \leq k, l \leq \infty$. The approximation results in Section 1.5 imply:

Theorem 4.1.7. *Let X and Y open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Then $\mathcal{C}^\infty_c(X) \otimes \mathcal{C}^\infty_c(Y)$ is sequentially dense in $\mathcal{C}^\infty_c(X \times Y)$.*

Proof. (see Sheet 7, Exercise 2).

4.2 Topologies on the tensor product of locally convex t.v.s.

Given two locally convex t.v.s. E and F, there various ways to construct a topology on the tensor product $E \otimes F$ which makes the vector space $E \otimes F$ in a t.v.s.. Indeed, starting from the topologies on E and F, one can define a topology on $E \otimes F$ either relying directly on the seminorms on E and F, or using an embedding of $E \otimes F$ in some space related to E and F over which a natural topology already exists. The first method leads to the so-called $\pi-$topology. The second method may lead instead to a variety of topologies, the most important of which is the so-called $\varepsilon-$topology that is based on the isomorphism between $E \otimes F$ and $B(E'_\sigma, F'_\sigma)$ (see Proposition **??**).

$\pi-$topology

Let us define the first main topology on $E \otimes F$ which we will see can be directly characterized by mean of the seminorms generating the topologies on the starting locally convex t.v.s. E and F.

Definition 4.2.1 ($\pi-$topology)**.**
Given two locally convex t.v.s. E and F, we define the $\pi-$topology *(or* projective topology*) on $E \otimes F$ to be the strongest locally convex topology on this vector space for which the canonical mapping $E \times F \to E \otimes F$ is continuous. The space $E \otimes F$ equipped with the $\pi-$topology will be denoted by $E \otimes_\pi F$.*

A basis of neighbourhoods of the origin in $E \otimes_\pi F$ is given by the family:

$$\mathcal{B} := \{conv_b(U_\alpha \otimes V_\beta) : U_\alpha \in \mathcal{B}_E,\, V_\beta \in \mathcal{B}_F\},$$

where $\mathcal{B}_E$ (resp. $\mathcal{B}_F$) is a basis of neighbourhoods of the origin in E (resp. in F), $U_\alpha \otimes V_\beta := \{x \otimes y \in E \otimes F : x \in U_\alpha, y \in V_\beta\}$ and $conv_b(U_\alpha \otimes V_\beta)$ denotes the smallest convex balanced subset of $E \otimes F$ containing $U_\alpha \otimes V_\beta$. In fact, on the one hand, the $\pi-$topology is by definition locally convex and so it has a basis $\mathcal{B}$ of convex balanced neighbourhoods of the origin in $E \otimes F$. Then, as the canonical mapping ϕ is continuous w.r.t. the $\pi-$topology, we have that for any $C \in \mathcal{B}$ there exist $U_\alpha \in \mathcal{B}_E$ and $V_\beta \in \mathcal{B}_F$ s.t. $U_\alpha \times V_\beta \subseteq \phi^{-1}(C)$. Hence, $U_\alpha \otimes V_\beta = \phi(U_\alpha \times V_\beta) \subseteq C$ and so $conv_b(U_\alpha \otimes V_\beta) \subseteq conv_b(C) = C$ which yields that the topology generated by $\mathcal{B}_\pi$ is finer than the $\pi-$topology. On the other hand, the canonical map ϕ is continuous w.r.t. the topology generated by $\mathcal{B}_\pi$, because for any $U_\alpha \in \mathcal{B}_E$ and $V_\beta \in \mathcal{B}_F$ we have that $\phi^{-1}(conv_b(U_\alpha \otimes V_\beta)) \supseteq \phi^{-1}(U_\alpha \otimes V_\beta) = U_\alpha \times V_\beta$ which is a neighbourhood of the origin in $E \times F$. Hence, the topology generated by $\mathcal{B}_\pi$ is coarser than the $\pi-$topology.

The $\pi-$topology on $E \otimes F$ can be described by means of the seminorms defining the locally convex topologies on E and F. Indeed, we have the following characterization of the $\pi-$topology.

Proposition 4.2.2. *Let E and F be two locally convex t.v.s. and let $\mathcal{P}$ (resp. $\mathcal{Q}$) be a family of seminorms generating the topology on E (resp. on F). The $\pi-$topology on $E \otimes F$ is generated by the family of seminorms*

$$\{p \otimes q : p \in \mathcal{P}, q \in \mathcal{Q}\},$$

where for any $p \in \mathcal{P}, q \in \mathcal{Q}, \theta \in E \otimes F$ we define:

$$(p \otimes q)(\theta) := \inf\{\rho > 0 : \theta \in \rho W\}$$

with

$$W := conv_b(U_p \otimes V_q), U_p := \{x \in E : p(x) \leq 1\}, \text{ and } V_q := \{y \in F : q(y) \leq 1\}.$$

Proof. (Sheet 7, Exercise 3) □

The seminorm $p \otimes q$ on $E \otimes F$ defined in the previous proposition is called *tensor product of the seminorms p and q* (or *projective cross seminorm*) and it can be represented in a more practical way that shows even more directly its relation to the seminorms defining the topologies on E and F.

Theorem 4.2.3.
a) *For any $\theta \in E \otimes F$, we have:*

$$(p\otimes q)(\theta) := \inf\left\{\sum_{k=1}^{r} p(x_k)q(y_k) : \theta = \sum_{k=1}^{r} x_k \otimes y_k, \, , x_k \in E, y_k \in F, r \in \mathbb{N}\right\}.$$

b) *For all $x \in E$ and $y \in F$, $(p \otimes q)(x \otimes y) = p(x)q(y)$.*

Proof.
a) As above, we set $U_p := \{x \in E : p(x) \leq 1\}$, $V_q := \{y \in F : q(y) \leq 1\}$ and $W := conv_b(U_p \otimes V_q)$. Let $\theta \in E \otimes F$. Let us preliminarily observe that the condition "$\theta \in \rho W$ for some $\rho > 0$" is equivalent to:

$$\theta = \sum_{k=1}^{N} t_k x_k \otimes y_k, \text{ with } \sum_{k=1}^{N} |t_k| \leq \rho, \, p(x_k) \leq 1, q(y_k) \leq 1, \forall k \in \{1, \dots, N\}.$$

If we set $\xi_k := t_k x_k$ and $\eta_k := y_k$, then

$$\theta = \sum_{k=1}^{N} \xi_k \otimes \eta_k \text{ with } \sum_{k=1}^{N} p(\xi_k)q(\eta_k) \leq \rho.$$

Then $\inf\{\sum_{k=1}^{r} p(x_k)q(y_k) : \theta = \sum_{k=1}^{r} x_k \otimes y_k, \, , x_k \in E, y_k \in F, r \in \mathbb{N}\} \leq \rho$ and since this is true for any $\rho > 0$ s.t. $\theta \in \rho W$ then we get:

$$\inf\left\{\sum_{i=1}^{r} p(x_i)q(y_i) : \theta = \sum_{i=1}^{r} x_i \otimes y_i, \, , x_i \in E, y_i \in F, r \in \mathbb{N}\right\} \leq (p \otimes q)(\theta).$$

Conversely, let us consider an arbitrary representation of θ, i.e.

$$\theta = \sum_{k=1}^{N} \xi_k \otimes \eta_k \text{ with } \xi_k \in E, \, \eta_k \in F,$$

and let $\rho > 0$ s.t. $\sum_{k=1}^{N} p(\xi_k)q(\eta_k) \leq \rho$. Let $\varepsilon > 0$. Define
- $I_1 := \{k \in \{1, \dots, N\} : p(\xi_k)q(\eta_k) \neq 0\}$
- $I_2 := \{k \in \{1, \dots, N\} : p(\xi_k) \neq 0 \text{ and } q(\eta_k) = 0\}$
- $I_3 := \{k \in \{1, \dots, N\} : p(\xi_k) = 0 \text{ and } q(\eta_k) \neq 0\}$
- $I_4 := \{k \in \{1, \dots, N\} : p(\xi_k) = 0 \text{ and } q(\eta_k) = 0\}$

and set
- $\forall k \in I_1$, $x_k := \frac{\xi_k}{p(\xi_k)}$, $y_k := \frac{\eta_k}{q(\eta_k)}$, $t_k := p(\xi_k)q(\eta_k)$
- $\forall k \in I_2$, $x_k := \frac{\xi_k}{p(\xi_k)}$, $y_k := \frac{N}{\varepsilon} p(\xi_k)\eta_k$, $t_k := \frac{\varepsilon}{N}$
- $\forall k \in I_3$, $x_k := \frac{N}{\varepsilon} q(\eta_k)\xi_k$, $y_k := \frac{\eta_k}{q(\eta_k)}$, $t_k := \frac{\varepsilon}{N}$

- $\forall k \in I_4$, $x_k := \frac{N}{\varepsilon}\xi_k$, $y_k := \eta_k$, $t_k := \frac{\varepsilon}{N}$

Then $\forall k \in \{1, \ldots, N\}$ we have that $p(x_k) \leq 1$ and $q(y_k) \leq 1$. Also we get:

$$\begin{aligned}\sum_{k=1}^{N} t_k x_k \otimes y_k &= \sum_{k\in I_1} p(\xi_k)q(\eta_k)\frac{\xi_k}{p(\xi_k)} \otimes \frac{\eta_k}{q(\eta_k)} + \sum_{k\in I_2} \frac{\varepsilon}{N}\frac{\xi_k}{p(\xi_k)} \otimes \frac{N}{\varepsilon}p(\xi_k)\eta_k \\ &+ \sum_{k\in I_3} \frac{\varepsilon}{N}\frac{N}{\varepsilon}q(\eta_k)\xi_k \otimes \frac{\eta_k}{q(\eta_k)} + \sum_{k\in I_4} \frac{\varepsilon}{N}\frac{N}{\varepsilon}\xi_k \otimes \eta_k \\ &= \sum_{k=1}^{N} \xi_k \otimes \eta_k = \theta\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{N} |t_k| &= \sum_{k\in I_1} p(\xi_k)q(\eta_k) + \sum_{k\in(I_2\cup I_3\cup I_4)} \frac{\varepsilon}{N} \\ &= \sum_{k\in I_1} p(\xi_k)q(\eta_k) + |I_2 \cup I_3 \cup I_4|\frac{\varepsilon}{N} \\ &\leq \sum_{k=1}^{n} p(\xi_k)q(\eta_k) + \varepsilon \leq \rho + \varepsilon.\end{aligned}$$

Hence, by our preliminary observation we get that $\theta \in (\rho+\varepsilon)W$. As this holds for any $\varepsilon > 0$, we have $\theta \in \rho W$. Therefore, we obtain that $(p \otimes q)(\theta) \leq \rho$ which yields

$$(p\otimes q)(\theta) \leq \inf\left\{\sum_{k=1}^{N} p(\xi_k)q(\eta_k) : \theta = \sum_{k=1}^{N} \xi_k \otimes \eta_k, \, , \xi_k \in E, \eta_k \in F, N \in \mathbb{N}\right\}.$$

b) Let $x \in E$ and $y \in F$. By using a), we immediately get that

$$(p \otimes q)(x \otimes y) \leq p(x)q(y).$$

Conversely, consider $M := span\{x\}$ and define $L : M \to \mathbb{K}$ as $L(\lambda x) := \lambda p(x)$ for all $\lambda \in \mathbb{K}$. Then clearly L is a linear functional on M and for any $m \in M$, i.e. $m = \lambda x$ for some $\lambda \in \mathbb{K}$, we have $|L(m)| = |\lambda|p(x) = p(\lambda x) = p(m)$. Therefore, Hahn-Banach theorem can be applied and provides that:

$$\exists\, x' \in E' \text{ s.t. } \langle x', x\rangle = p(x) \text{ and } |\langle x', x_1\rangle| \leq p(x_1), \forall\, x_1 \in E. \tag{4.2}$$

Repeating this reasoning for y we get that:

$$\exists\, y' \in F' \text{ s.t. } \langle y', y\rangle = q(y) \text{ and } |\langle y', y_1\rangle| \leq q(y_1), \forall\, y_1 \in F. \tag{4.3}$$

Let us consider now any representation of $x\otimes y$, namely $x\otimes y=\sum_{k=1}^{N} x_k\otimes y_k$ with $x_k\in E$, $y_k\in F$ and $N\in\mathbb{N}$. Then using the second part of (4.2) and (4.3) we obtain:

$$\left|\langle x'\otimes y', x\otimes y\rangle\right| \leq \sum_{k=1}^{N} |\langle x', x_k\rangle|\cdot|\langle y', y_k\rangle| \leq \sum_{k=1}^{N} p(x_k)q(x_k).$$

Since this is true for any representation of $x\otimes y$, we deduce by a) that:

$$\left|\langle x'\otimes y', x\otimes y\rangle\right| \leq (p\otimes q)(x\otimes y).$$

The latter together with the first part of (4.2) and (4.3) gives:

$$p(x)q(y)=|p(x)|\cdot|q(y)|=|\langle x', x\rangle|\cdot|\langle y', y\rangle|=\left|\langle x'\otimes y', x\otimes y\rangle\right|\leq (p\otimes q)(x\otimes y).$$

□

Proposition 4.2.4. *Let E and F be two locally convex t.v.s.. $E\otimes_\pi F$ is Hausdorff if and only if E and F are both Hausdorff.*

Proof. (Sheet 7, Exercise 4) □

Corollary 4.2.5. *Let (E,p) and (F,q) be seminormed spaces. Then $p\otimes q$ is a norm on $E\otimes F$ if and only if p and q are both norms.*

Proof.
Under our assumptions, the $\pi-$topology on $E\otimes F$ is generated by the single seminorm $p\otimes q$. Then, recalling that a seminormed space is normed iff it is Hausdorff and using Proposition 4.2.4, we get: $(E\otimes F, p\otimes q)$ is normed $\Leftrightarrow$ $E\otimes_\pi F$ is Hausdorff $\Leftrightarrow$ E and F are both Hausdorff $\Leftrightarrow$ (E,p) and (F,q) are both normed. □

Definition 4.2.6. *Let (E,p) and (F,q) be normed spaces. The normed space $(E\otimes F, p\otimes q)$ is called the* projective tensor product *of E and F and $p\otimes q$ is said to be the corresponding* projective tensor norm.

In analogy with the algebraic case (see Theorem 4.1.4-b), we also have a universal property for the space $E\otimes_\pi F$.

Proposition 4.2.7.
Let E, F be locally convex spaces. The $\pi-$topology on $E\otimes_\pi F$ is the unique locally convex topology on $E\otimes F$ such that the following property holds:

(UP) *For every locally convex space G, the algebraic isomorphism between the space of bilinear mappings from $E \times F$ into G and the space of all linear mappings from $E \otimes F$ into G (given by Theorem 4.1.4-b) induces an algebraic isomorphism between $B(E,F;G)$ and $L(E \otimes F;G)$, where $B(E,F;G)$ denotes the space of all continuous bilinear mappings from $E \times F$ into G and $L(E \otimes F;G)$ the space of all continuous linear mappings from $E \otimes F$ into G.*

Proof. Let τ be a locally convex topology on $E \otimes F$ such that the property (UP) holds. Then (UP) holds in particular for $G = (E \otimes F, \tau)$. Therefore, since in the algebraic isomorphism given by Theorem 4.1.4-b) in this case the canonical mapping $\phi : E \times F \to E \otimes F$ corresponds to the identity $id : E \otimes F \to E \otimes F$, we get that $\phi : E \times F \to E \otimes_\tau F$ has to be continuous.

$$\begin{array}{ccc} E \times F & \xrightarrow{\phi} & E \otimes_\tau F \\ \downarrow \phi & \nearrow id & \\ E \otimes_\tau F & & \end{array}$$

This implies that $\tau \subseteq \pi$ by definition of $\pi-$topology. On the other hand, (UP) also holds for $G = (E \otimes F, \pi)$.

$$\begin{array}{ccc} E \times F & \xrightarrow{\phi} & E \otimes_\pi F \\ \downarrow \phi & \nearrow id & \\ E \otimes_\tau F & & \end{array}$$

Hence, since by definition of $\pi-$topology $\phi : E \times F \to E \otimes_\pi F$ is continuous, the $id : E \otimes_\tau F \to E \otimes_\pi F$ has to be also continuous. This means that $\pi \subseteq \tau$, which completes the proof. □

Corollary 4.2.8. $(E \otimes_\pi F)' \cong B(E,F)$.

Proof. By taking $G = \mathbb{K}$ in Proposition 4.2.7, we get the conclusion. □