(UP) *For every locally convex space G, the algebraic isomorphism between the space of bilinear mappings from $E \times F$ into G and the space of all linear mappings from $E \otimes F$ into G (given by Theorem 4.1.4-b) induces an algebraic isomorphism between $B(E, F; G)$ and $L(E \otimes F; G)$, where $B(E, F; G)$ denotes the space of all continuous bilinear mappings from $E \times F$ into G and $L(E \otimes F; G)$ the space of all continuous linear mappings from $E \otimes F$ into G.*

Proof. Let τ be a locally convex topology on $E \otimes F$ such that the property (UP) holds. Then (UP) holds in particular for $G = (E \otimes F, \tau)$. Therefore, since in the algebraic isomorphism given by Theorem 4.1.4-b) in this case the canonical mapping $\phi : E \times F \to E \otimes F$ corresponds to the identity $id : E \otimes F \to E \otimes F$, we get that $\phi : E \times F \to E \otimes_\tau F$ has to be continuous.

$$\begin{array}{ccc} E \times F & \xrightarrow{\phi} & E \otimes_\tau F \\ \downarrow{\scriptstyle\phi} & \nearrow{\scriptstyle id} & \\ E \otimes_\tau F & & \end{array}$$

This implies that $\tau \subseteq \pi$ by definition of $\pi-$topology. On the other hand, (UP) also holds for $G = (E \otimes F, \pi)$.

$$\begin{array}{ccc} E \times F & \xrightarrow{\phi} & E \otimes_\pi F \\ \downarrow{\scriptstyle\phi} & \nearrow{\scriptstyle id} & \\ E \otimes_\tau F & & \end{array}$$

Hence, since by definition of $\pi-$topology $\phi : E \times F \to E \otimes_\pi F$ is continuous, the $id : E \otimes_\tau F \to E \otimes_\pi F$ has to be also continuous. This means that $\pi \subseteq \tau$, which completes the proof. □

Corollary 4.2.8. $(E \otimes_\pi F)' \cong B(E, F)$.

Proof. By taking $G = \mathbb{K}$ in Proposition 4.2.7, we get the conclusion. □

4.2.2 Tensor product t.v.s. and bilinear forms

Before introducing the $\varepsilon-$topology, let us present the above mentioned algebraic isomorphism between the tensor product of two locally convex t.v.s. and the spaces of bilinear forms on the product of their weak duals. Since we are going to deal with topological duals of t.v.s., in this section we will always assume that E and F are two non-trivial locally convex t.v.s. over the same field $\mathbb{K}$ with non-trivial topological duals. Let G be another t.v.s. over $\mathbb{K}$ and $\phi : E \times F \to G$ a bilinear map. The bilinear map ϕ is said to be *separately continuous* if for all $x_0 \in E$ and for all $y_0 \in F$ the following two linear mappings are continuous:

$$\begin{array}{rccl} \phi_{x_0}: & F & \to & G \\ & y & \to & \phi(x_0, y) \end{array} \qquad \text{and} \qquad \begin{array}{rccl} \phi_{y_0}: & E & \to & G \\ & x & \to & \phi(x, y_0). \end{array}$$

We denote by $\mathcal{B}(E, F, G)$ the linear space of all separately continuous bilinear maps from $E \times F$ into G and by $B(E, F, G)$ its linear subspace of all continuous bilinear maps from $E \times F$ into G. When $G = \mathbb{K}$ we write $\mathcal{B}(E, F)$ and $B(E, F)$, respectively. Note that $B(E, F, G) \subset \mathcal{B}(E, F, G)$, i.e. any continuous bilinear map is separately continuous but the converse does not hold in general.

The following proposition describes an important relation existing between tensor products and bilinear forms.

Proposition 4.2.9. *Let E and F be non-trivial locally convex t.v.s. over $\mathbb{K}$ with non-trivial topological duals. The space $B(E'_\sigma, F'_\sigma)$ is a tensor product of E and F.*

Recall that E'_σ (resp. F'_σ) denotes the topological dual E' of E (resp. F' of F) endowed with the weak topology defined in Section 3.2.

Proof.
Let us consider the bilinear mapping:

$$\begin{array}{rccl} \phi: & E \times F & \to & B(E'_\sigma, F'_\sigma) \\ & (x, y) & \mapsto & \begin{array}{rccl} \phi(x, y): & E'_\sigma \times F'_\sigma & \to & \mathbb{K} \\ & (x', y') & \mapsto & \langle x', x\rangle\langle y', y\rangle. \end{array} \end{array} \tag{4.4}$$

We first show that E and F are ϕ-linearly disjoint. Let $r, s \in \mathbb{N}$, $x_1, \ldots, x_r$ be linearly independent in E and $y_1, \ldots, y_s$ be linearly independent in F. In their correspondence, select[1] $x'_1, \ldots, x'_r \in E'$ and $y'_1, \ldots, y'_s \in F'$ such that

$$\langle x'_m, x_j\rangle = \delta_{mj}, \, \forall\, m, j \in \{1, \ldots, r\} \;\text{ and }\; \langle y'_n, y_k\rangle = \delta_{nk} \, \forall\, n, k \in \{1, \ldots, s\}.$$

Then we have that:

$$\phi(x_j, y_k)(x'_m, y'_n) = \langle x'_m, x_j\rangle\langle y'_n, y_k\rangle = \begin{cases} 1 & \text{if } m = j \text{ and } n = k \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

[1] This can be done using Lemma 3.2.9 together with the assumption that E' and F' are not trivial.

This implies that the set $\{\phi(x_j, y_k) : j = 1, \dots, r,\ k = 1, \dots, s\}$ consists of linearly independent elements. Indeed, if there exists $\lambda_{jk} \in \mathbb{K}$ s.t.

$$\sum_{j=1}^{r} \sum_{k=1}^{s} \lambda_{jk} \phi(x_j, y_k) = 0$$

then for all $m \in \{1, \dots, r\}$ and all $n \in \{1, \dots, r\}$ we have that:

$$\sum_{j=1}^{r} \sum_{k=1}^{s} \lambda_{jk} \phi(x_j, y_k)(x'_m, y'_n) = 0$$

and so by using (4.5) that all $\lambda_{mn} = 0$.

We have therefore showed that (LD') holds and so, by Proposition 4.1.2, E and F are ϕ-linearly disjoint. Let us briefly sketch the main steps of the proof that $span(\phi(E \times F)) = B(E'_\sigma, F'_\sigma)$.

a) Take any $\varphi \in B(E'_\sigma, F'_\sigma)$. By the continuity of φ, it follows that there exist finite subsets $A \subset E$ and $B \subset F$ s.t. $|\varphi(x', y')| \leq 1$, $\forall\, x' \in A^\circ$, $\forall\, y' \in B^\circ$.

b) Set $E_A := span(A)$ and $F_B := span(B)$. Since E_A and E_B are finite dimensional, their orthogonals $(E_A)^\circ$ and $(F_B)^\circ$ have finite codimension and so

$$E' \times F' = (M' \oplus (E_A)^\circ) \times (N' \oplus (F_B)^\circ) = (M' \times N') \oplus ((E_A)^\circ \times F') \oplus (E' \times (F_B)^\circ),$$

where M' and N' finite dimensional subspaces of E' and F', respectively.

c) Using a) and b) one can prove that φ vanishes on the direct sum $((E_A)^\circ \times F') \oplus (E' \times (F_B)^\circ)$ and so that φ is completely determined by its restriction to a finite dimensional subspace $M' \times N'$ of $E' \times F'$.

d) Let $r := dim(E_A)$ and $s := dim(F_B)$. Then there exist $x_1, \dots, x_r \in E_A$ and $y_1, \dots, y_s \in F_B$ s.t. the restriction of φ to $M' \times N'$ is given by

$$(x', y') \mapsto \sum_{i=1}^{r} \sum_{j=1}^{s} \langle x', x_i \rangle \langle y', y_j \rangle.$$

Hence, by c), we can conclude that $\phi \in span(\phi(E \times F))$.

□

4.2.3 $\varepsilon-$topology

In order to define the $\varepsilon-$topology on $E \otimes F$, we need to introduce the so-called *topology of bi-equicontinuous convergence* on the space $B(E'_\sigma, F'_\sigma)$. To this aim we first need to study a bit the notion of *equicontinuous sets of mappings* between t.v.s..

Definition 4.2.10. *Let X and Y be two t.v.s.. A set S of linear mappings of X into Y is said to be* equicontinuous *if for any neighbourhood V of the origin in Y there exists a neighbourhood U of the origin in X such that*

$$\forall\, f \in S,\, x \in U \;\Rightarrow\; f(x) \in V$$

i.e.

$$\forall\, f \in S,\, f(U) \subseteq V \quad (\textit{or } U \subseteq f^{-1}(V)).$$

The equicontinuity condition can be also rewritten as follows: S is equicontinuous if for any neighbourhood V of the origin in Y there exists a neighbourhood U of the origin in X such that $\bigcup_{f\in S} f(U) \subseteq V$ or, equivalently, if for any neighbourhood V of the origin in Y the set $\bigcap_{f\in S} f^{-1}(V)$ is a neighbourhood of the origin in X.

Note that if S is equicontinuous then each mapping $f \in S$ is continuous but clearly the converse does not hold.

A first property of equicontinuous sets which is clear from the definition is that any subset of an equicontinuous set is itself equicontinuous. We are going now to introduce now few more properties of equicontinuous sets of linear functionals on a t.v.s. which will be useful in the following.

Proposition 4.2.11. *A set of continuous linear functionals on a t.v.s. X is equicontinuous if and only if it is contained in the polar of some neighbourhood of the origin in X.*

Proof.
For any $\rho > 0$, let us denote by $D_\rho := \{k \in \mathbb{K} : |k| \leq \rho\}$. Let H be an equicontinuous set of linear forms on X. Then there exists a neighbourhood U of the origin in X s.t. $\bigcup_{f\in H} f(U) \subseteq D_1$, i.e. $\forall f \in H$, $|\langle f, x\rangle| \leq 1, \forall\, x \in U$, which means exactly that $H \subseteq U^\circ$.

Conversely, let U be an arbitrary neighbourhood of the origin in X and let us consider the polar $U^\circ := \{f \in X' : \sup_{x\in U} |\langle f, x\rangle| \leq 1\}$. Then for any $\rho > 0$

$$\forall\, f \in U^\circ,\, |\langle f, y\rangle| \leq \rho,\, \forall\, y \in \rho U,$$

which is equivalent to

$$\bigcup_{f\in U^\circ} f(\rho U) \subseteq D_\rho.$$

This means that U° is equicontinuous and so any subset H of U° is also equicontinuous, which yields the conclusion. □

Proposition 4.2.12. *Let X be a locally convex Hausdorff t.v.s.. Any equicontinuous subset of X' is bounded in X'_σ.*

Proof. Let H be an equicontinuous subset of X'. Then, by Proposition 4.2.11, we get that there exists a neighbourhood U of the origin in X such that $H \subseteq U^\circ$. By Banach-Alaoglu theorem (see Theorem 3.3.3), we know that $U^\circ$is compact in X'_σ and so bounded by Proposition 2.2.4. Hence, by Proposition 2.2.2-4, H is also bounded in X'_σ. □

It is also possible to show, but we are not going to prove this here, that the following holds.

Proposition 4.2.13. *Let X be a locally convex Hausdorff t.v.s.. The union of all equicontinuous subsets of X' is dense in X'_σ.*

Let us now turn our attention to the space $B(X,Y;Z)$ of continuous bilinear mappings from $X \times Y$ to Z, when X, Y and Z are three locally convex t.v.s.. There is a natural way of introducing a topology on this space which is a kind of generalization to what we have done when we defined polar topologies in Chapter 3.

Consider a family Σ (resp. Γ) of bounded subsets of X (resp. Y) satisfying the following properties:
(P1) If $A_1, A_2 \in \Sigma$, then $\exists\, A_3 \in \Sigma$ s.t. $A_1 \cup A_2 \subseteq A_3$.
(P2) If $A_1 \in \Sigma$ and $\lambda \in \mathbb{K}$, then $\exists\, A_2 \in \Sigma$ s.t. $\lambda A_1 \subseteq A_2$.
(resp. satisfying (P1) and (P2) replacing Σ by Γ). The Σ-Γ-topology on $B(X,Y;Z)$, or topology of uniform convergence on subsets of the form $A \times B$ with $A \in \Sigma$ and $B \in \Gamma$, is defined by taking as a basis of neighbourhoods of the origin in $B(X,Y;Z)$ the following family:

$$\mathcal{U} := \{\mathcal{U}(A,B;W) :\ A \in \Sigma, B \in \Gamma, W \in \mathcal{B}_Z(o)\}$$

where

$$\mathcal{U}(A,B;W) := \{\varphi \in B(X,Y;Z) : \varphi(A,B) \subseteq W\}$$

and $\mathcal{B}_Z(o)$ is a basis of neighbourhoods of the origin in Z. It is not difficult to verify that (c.f. [5, Chapter 32]):

a) each $\mathcal{U}(A,B;W)$ is an absorbing, convex, balanced subset of $B(X,Y;Z)$;
b) the Σ-Γ-topology makes $B(X,Y;Z)$ into a locally convex t.v.s. (by Theorem 4.1.14 of TVS-I);

c) If Z is Hausdorff, the union of all subsets in Σ is dense in X and the of all subsets in Γ is dense in Y, then the Σ-Γ-topology on $B(X, Y; Z)$ is Hausdorff.

In particular, given two locally convex Hausdorff t.v.s. E and F, we call *bi-equicontinuous topology on* $B(E'_\sigma, F'_\sigma)$ the Σ-Γ-topology when Σ is the family of all equicontinuous subsets of E' and Γ is the family of all equicontinuous subsets of F'. Note that we can make this choice of Σ and Γ, because by Proposition 4.2.12 all equicontinuous subsets of E' (resp. F') are bounded in E'_σ (resp. F'_σ) and satisfy the properties $(P1)$ and $(P2)$. A basis for the bi-equicontinuous topology $B(E'_\sigma, F'_\sigma)$ is then given by:

$$\mathcal{U} := \{\mathcal{U}(A, B; \varepsilon) : A \in \Sigma, B \in \Gamma, \varepsilon > 0\}$$

where

$$\begin{aligned}\mathcal{U}(A, B; \varepsilon) &:= \{\varphi \in B(E'_\sigma, F'_\sigma) : \varphi(A, B) \subseteq D_\varepsilon\} \\ &= \{\varphi \in B(E'_\sigma, F'_\sigma) : |\varphi(x', y')| \leq \varepsilon, \forall x' \in A, \forall y' \in B\}\end{aligned}$$

and $D_\varepsilon := \{k \in \mathbb{K} : |k| \leq \varepsilon\}$. By using a) and b), we get that $B(E'_\sigma, F'_\sigma)$ endowed with the bi-equicontinuous topology is a locally convex t.v.s.. Also, by using Proposition 4.2.13 together with c), we can prove that the bi-equicontinuous topology on $B(E'_\sigma, F'_\sigma)$ is Hausdorff (as E and F are both assumed to be Hausdorff).

We can then use the isomorphism between $E \otimes F$ and $B(E'_\sigma, F'_\sigma)$ provided by Proposition 4.2.9[2] to carry the bi-equicontinuous topology on $B(E'_\sigma, F'_\sigma)$ over $E \otimes_\varepsilon F$.

Definition 4.2.14 ($\varepsilon-$topology)**.**
*Given two locally convex Hausdorff t.v.s. E and F, we define the $\varepsilon-$*topology *on $E \otimes F$ to the topology carried over (from $B(E'_\sigma, F'_\sigma)$ endowed with the bi-equicontinuous topology, i.e. topology of uniform convergence on the products of an equicontinuous subset of E' and an equicontinuous subset of F'. The space $E \otimes F$ equipped with the $\varepsilon-$topology will be denoted by $E \otimes_\varepsilon F$.*

It is clear then $E \otimes_\varepsilon F$ is a locally convex Hausdorff t.v.s.. Moreover, we have that:

Proposition 4.2.15. *Given two locally convex Hausdorff t.v.s. E and F, the canonical mapping from $E \times F$ into $E \otimes_\varepsilon F$ is continuous. Hence, the $\pi-$topology is finer than the $\varepsilon-$topology on $E \otimes F$.*

[2] Recall that non-trivial locally convex Hausdorff t.v.s. have non-trivial topological dual by Proposition 3.2.7

Proof.
By definition of $\varepsilon-$topology, it is enough to show that the canonical mapping ϕ from $E \times F$ into $B(E'_\sigma, F'_\sigma)$ defined in (4.4) is continuous w.r.t. the bi-equicontinuous topology on $B(E'_\sigma, F'_\sigma)$. Let $\varepsilon > 0$, A any equicontinuous subset of E' and B any equicontinuous subset of F', then by Proposition 4.2.11 we get that there exist a neighbourhood N_A of the origin in E and a neighbourhood N_B of the origin in F s.t. $A \subseteq (N_A)^\circ$ and $B \subseteq (N_B)^\circ$. Hence, we obtain that

$$\begin{aligned}
\phi^{-1}(\mathcal{U}(A,B;\varepsilon)) &= \{(x,y) \in E \times F : \phi(x,y) \in \mathcal{U}(A,B;\varepsilon)\}\\
&= \big\{(x,y) \in E \times F : |\phi(x,y)(x',y')| \leq \varepsilon, \forall\, x' \in A, \forall\, y' \in B\big\}\\
&= \big\{(x,y) \in E \times F : |\langle x', x\rangle\langle y', y\rangle| \leq \varepsilon, \forall\, x' \in A, \forall\, y' \in B\big\}\\
&\supseteq \big\{(x,y) \in E \times F : |\langle x', x\rangle\langle y', y\rangle| \leq \varepsilon, \forall\, x' \in (N_A)^\circ, \forall\, y' \in (N_B)^\circ\big\}\\
&\supseteq \varepsilon N_A \times N_B,
\end{aligned}$$

which proves the continuity of ϕ as $\varepsilon N_A \times N_B$ is a neighbourhood of the origin in $E \times F$. □