Let us first recall some standard notations. For any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ one defines $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. For any $\beta \in \mathbb{N}_0^d$, the symbol $D^\beta$ denotes the partial derivative of order $|\beta|$ where $|\beta| := \sum_{i=1}^d \beta_i$, i.e.

$$D^\beta := \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}} = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_d}}{\partial x_d^{\beta_d}}.$$

**Example: $\mathcal{C}^s(\Omega)$ with $\Omega \subseteq \mathbb{R}^d$ open.**

Let $\Omega \subseteq \mathbb{R}^d$ open in the euclidean topology. For any $s \in \mathbb{N}_0$, we denote by $\mathcal{C}^s(\Omega)$ the set of all real valued *$s-$times continuously differentiable functions* on $\Omega$, i.e. all the derivatives of order $\leq s$ exist (at every point of $\Omega$) and are continuous functions in $\Omega$. Clearly, when $s = 0$ we get the set $\mathcal{C}(\Omega)$ of all real valued continuous functions on $\Omega$ and when $s = \infty$ we get the so-called set of all *infinitely differentiable functions* or *smooth functions* on $\Omega$. For any $s \in \mathbb{N}_0$, $\mathcal{C}^s(\Omega)$ (with pointwise addition and scalar multiplication) is a vector space over $\mathbb{R}$.

Let us consider the following family $\mathcal{P}$ of seminorms on $\mathcal{C}^s(\Omega)$:

$$p_{m,K}(f) := \sup_{\substack{\beta \in \mathbb{N}_0^d \\ |\beta| \leq m}} \sup_{x \in K} \left| (D^\beta f)(x) \right|, \forall K \subset \Omega \text{ compact}, \forall m \in \{0, 1, \dots, s\},$$

(Note when $s = \infty$ we have $m \in \mathbb{N}_0$.) The topology $\tau_{\mathcal{P}}$ generated by $\mathcal{P}$ is usually referred as *$\mathcal{C}^s$-topology* or *topology of uniform convergence on compact sets of the functions and their derivatives up to order $s$.*

**1)** The $\mathcal{C}^s$-topology clearly turns $\mathcal{C}^s(\Omega)$ into a locally convex t.v.s., which is evidently Hausdorff as the family $\mathcal{P}$ is separating (see Prop 4.3.3 TVS-I). Indeed, if $p_{m,K}(f) = 0$, $\forall m \in \{0, 1, \dots, s\}$ and $\forall K$ compact subset of $\Omega$ then in particular $p_{0,\{x\}}(f) = |f(x)| = 0$ $\forall x \in \Omega$, which implies $f \equiv 0$ on $\Omega$.

**2)** $(\mathcal{C}^s(\Omega), \tau_{\mathcal{P}})$ is metrizable.

By Proposition 1.1.5, this is equivalent to prove that the $\mathcal{C}^s$-topology can be generated by a countable separating family of seminorms. In order to show this, let us first observe that for any two non-negative integers $m_1 \leq m_2 \leq s$ and any two compact $K_1 \subseteq K_2 \subset \Omega$ we have:

$$p_{m_1,K_1}(f) \leq p_{m_2,K_2}(f), \quad \forall f \in \mathcal{C}^s(\Omega).$$

Then the family $\{p_{s,K} : K \subset \Omega \text{ compact}\}$ generates the $\mathcal{C}^s-$topology on $\mathcal{C}^s(\Omega)$. Moreover, it is easy to show that there is a sequence of compact subsets $\{K_j\}_{j \in \mathbb{N}}$ of $\Omega$ such that $K_j \subseteq \mathring{K}_{j+1}$ for all $j \in \mathbb{N}$ and $\Omega = \cup_{j \in \mathbb{N}} K_j$. Then for any $K \subset \Omega$ compact we have that there exists $j \in \mathbb{N}$ s.t. $K \subseteq K_j$ and so $p_{s,K}(f) \leq p_{s,K_j}(f)$, $\forall f \in \mathcal{C}^s(\Omega)$. Hence, the countable family of seminorms $\{p_{s,K_j} : j \in \mathbb{N}\}$ generates the $\mathcal{C}^s-$topology on $\mathcal{C}^s(\Omega)$ and it is separating. Indeed, if $p_{s,K_j}(f) = 0$ for all $j \in \mathbb{N}$ then for every $x \in \Omega$ we have $x \in K_i$ for some $i \in \mathbb{N}$ and so $0 \leq |f(x)| \leq p_{s,K_i}(f) = 0$, which implies $|f(x)| = 0$ for all $x \in \Omega$, i.e. $f \equiv 0$ on $\Omega$.

**3)** $(\mathcal{C}^s(\Omega), \tau_{\mathcal{P}})$ is complete.
By Proposition 1.1.6, it is enough to show that it is sequentially complete. Let $(f_\nu)_{\nu\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{C}^k(\Omega)$, i.e.

$$\forall m \leq s, \forall K \subset \Omega \text{ compact}, \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall \mu, \nu \geq N : p_{m,K}(f_\nu - f_\mu) \leq \varepsilon. \tag{1.7}$$

In particular, for any $x \in \Omega$ by taking $m = 0$ and $K = \{x\}$ we get that the sequence $(f_\nu(x))_{\nu\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Hence, by the completeness of $\mathbb{R}$, it has a limit point in $\mathbb{R}$ which we denote by $f(x)$. Obviously $x \mapsto f(x)$ is a function on $\Omega$, so we have just showed that the sequence $(f_\nu)_{\nu\in\mathbb{N}}$ converge to $f$ pointwise in $\Omega$, i.e.

$$\forall x \in \Omega, \forall \varepsilon > 0, \exists M_x \in \mathbb{N} \text{ s.t. } \forall \mu \geq M_x : |f_\mu(x) - f(x)| \leq \varepsilon. \tag{1.8}$$

Then it is easy to see that $(f_\nu)_{\nu\in\mathbb{N}}$ converges uniformly to $f$ in every compact subset $K$ of $\Omega$. Indeed, we get it just passing to the pointwise limit for $\mu \to \infty$ in (1.7) for $m = 0$. [2]

As $(f_\nu)_{\nu\in\mathbb{N}}$ converges uniformly to $f$ in every compact subset $K$ of $\Omega$, by taking this subset identical with a suitable neighbourhood of any point of $\Omega$, we conclude by Lemma 1.2.2 that $f$ is continuous in $\Omega$.

- If $s = 0$, this completes the proof since we just showed $f_\nu \to f$ in the $\mathcal{C}^0-$topology and $f \in \mathcal{C}(\Omega)$.
- If $0 < s < \infty$, then observe that since $(f_\nu)_{\nu\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}^s(\Omega)$, for each $j \in \{1, \dots, d\}$ the sequence $(\frac{\partial}{\partial x_j} f_\nu)_{\nu\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}^{s-1}(\Omega)$. Then induction on $s$ allows us to conclude that, for each $j \in \{1, \dots, d\}$, the $(\frac{\partial}{\partial x_j} f_\nu)_{\nu\in\mathbb{N}}$ converges uniformly on every compact subset of $\Omega$ to a function $g^{(j)} \in \mathcal{C}^{s-1}(\Omega)$ and by Lemma 1.2.3 we have that $g^{(j)} = \frac{\partial}{\partial x_j} f$. Hence, we have showed that $(f_\nu)_{\nu\in\mathbb{N}}$ converges to $f$ in the $\mathcal{C}^s-$topology with $f \in \mathcal{C}^s(\Omega)$.

[2] Detailed proof: Let $\varepsilon > 0$. By (1.7) for $m = 0$, $\exists N \in \mathbb{N}$ s.t.$\forall \mu, \nu \geq N : |f_\nu(x) - f_\mu(x)| \leq \frac{\varepsilon}{2}, \forall x \in K$. Now for each fixed $x \in K$ one can always choose a $\mu_x$ larger than both $N$ and the corresponding $M_x$ as in (1.8) so that $|f_{\mu_x}(x) - f(x)| \leq \frac{\varepsilon}{2}$. Hence, for all $\nu \geq N$ one gets that $|f_\nu(x) - f(x)| \leq |f_\nu(x) - f_{\mu_x}(x)| + |f_{\mu_x}(x) - f(x)| \leq \varepsilon, \forall x \in K$

- If $s = \infty$, then we are also done by the definition of the $\mathcal{C}^\infty$-topology. Indeed, a Cauchy sequence $(f_\nu)_{\nu\in\mathbb{N}}$ in $\mathcal{C}^\infty(\Omega)$ it is in particular a Cauchy sequence in the subspace topology given by $\mathcal{C}^s(\Omega)$ for any $s \in \mathbb{N}$ and hence, for what we have already showed, it converges to $f \in \mathcal{C}^s(\Omega)$ in the $C^s-$topology for any $s \in \mathbb{N}$. This means exactly that $(f_\nu)_{\nu\in\mathbb{N}}$ converges to $f \in \mathcal{C}^\infty(\Omega)$ in the in $\mathcal{C}^\infty-$topology.

Let us prove now the two lemmas which we have used in the previous proof:

**Lemma 1.2.2.** *Let $A \subset \mathbb{R}^d$ and $(f_\nu)_{\nu\in\mathbb{N}}$ in $\mathcal{C}(A)$. If $(f_\nu)_{\nu\in\mathbb{N}}$ converges to a function $f$ uniformly in $A$ then $f \in \mathcal{C}(A)$.*

*Proof.*
Let $x_0 \in A$ and $\varepsilon > 0$. By the uniform convergence of $(f_\nu)_{\nu\in\mathbb{N}}$ to $f$ in $A$ we get that:

$$\exists N \in \mathbb{N} \text{ s.t. } \forall \nu \geq N : |f_\nu(y) - f(y)| \leq \frac{\varepsilon}{3}, \forall y \in A.$$

Fix such a $\nu$. As $f_\nu$ is continuous on $A$ then:

$$\exists \delta > 0 \text{ s.t. } \forall x \in A \text{ with } |x - x_0| \leq \delta \text{ we have } |f_\nu(x) - f_\nu(x_0)| \leq \frac{\varepsilon}{3}.$$

Therefore, we obtain that $\forall x \in A$ with $|x - x_0| \leq \delta$ :

$$|f(x) - f(x_0)| \leq |f(x) - f_\nu(x)| + |f_\nu(x) - f_\nu(x_0)| + |f_\nu(x_0) - f(x_0)| \leq \varepsilon.$$

□

**Lemma 1.2.3.** *Let $A \subset \mathbb{R}^d$ and $(f_\nu)_{\nu\in\mathbb{N}}$ in $\mathcal{C}^1(A)$. If $(f_\nu)_{\nu\in\mathbb{N}}$ converges to a function $f$ uniformly in $A$ and for each $j \in \{1, \ldots, d\}$ the sequence $(\frac{\partial}{\partial x_j} f_\nu)_{\nu\in\mathbb{N}}$ converges to a function $g^{(j)}$ uniformly in $A$, then*

$$g^{(j)} = \frac{\partial}{\partial x_j} f, \forall j \in \{1, \ldots, d\}.$$

*This means in particular that $f \in \mathcal{C}^1(A)$.*

*Proof.* (for $d = 1$, $A = [a, b]$)
By the fundamental theorem of calculus, we have that for any $x \in A$

$$f_\nu(x) - f_\nu(a) = \int_a^x \frac{\partial}{\partial t} f_\nu(t) dt. \tag{1.9}$$

By the uniform convergence of the first derivatives to $g^{(1)}$ and by the Lebesgue dominated convergence theorem, we also have

$$\int_a^x \frac{\partial}{\partial t} f_\nu(t)dt \to \int_a^x g^{(1)}(t)dt, \text{ as } \nu \to \infty. \tag{1.10}$$

Using (1.9) and (1.10) together with the assumption that $f_\nu \to f$ unformly in $A$, we obtain that:

$$f(x) - f(a) = \int_a^x g^{(1)}(t)dt,$$

i.e. $\left(\frac{\partial}{\partial x} f\right)(x) = g^{(1)}(x), \forall x \in A.$ □

**Example: The Schwarz space $S(\mathbb{R}^d)$.**

The *Schwartz space* or *space of rapidly decreasing functions* on $\mathbb{R}^d$ is defined as the set $\mathcal{S}(\mathbb{R}^d)$ of all real-valued functions which are defined and infinitely differentiable on $\mathbb{R}^d$ and which have the additional property (regulating their growth at infinity) that all their derivatives tend to zero at infinity faster than any inverse power of $x$, i.e.

$$\mathcal{S}(\mathbb{R}^d) := \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} \left| x^\alpha (D^\beta f)(x) \right| < \infty, \ \forall \alpha, \beta \in \mathbb{N}_0^d \right\}.$$

(For example, any smooth function $f$ with compact support in $\mathbb{R}^d$ is in $\mathcal{S}(\mathbb{R}^d)$, since any derivative of $f$ is continuous and supported on a compact subset of $\mathbb{R}^d$, so $x^\alpha(D^\beta f(x))$ has a maximum in $\mathbb{R}^d$ by the extreme value theorem.)

The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is a vector space over $\mathbb{R}$ and we equip it with the topology $\tau_\mathcal{Q}$ given by the family $\mathcal{Q}$ of seminorms on $\mathcal{S}(\mathbb{R}^d)$:

$$q_{m,k}(f) := \sup_{\substack{\beta \in \mathbb{N}_0^d \\ |\beta| \leq m}} \sup_{x \in \mathbb{R}^d} (1 + |x|)^k \left| (D^\beta) f(x) \right|, \ \forall m, k \in \mathbb{N}_0.$$

Note that $f \in \mathcal{S}(\mathbb{R}^d)$ if and only if $\forall m, k \in \mathbb{N}_0, q_{m,k}(f) < \infty$.

The space $\mathcal{S}(\mathbb{R}^d)$ is a linear subspace of $\mathcal{C}^\infty(\mathbb{R}^d)$, but $\tau_\mathcal{Q}$ is finer than the subspace topology induced on it by $\tau_\mathcal{P}$ where $\mathcal{P}$ is the family of seminorms defined on $\mathcal{C}^\infty(\mathbb{R}^d)$ as in the above example. Indeed, it is clear that for any $f \in \mathcal{S}(\mathbb{R}^d)$, any $m \in \mathbb{N}_0$ and any $K \subset \mathbb{R}^d$ compact we have $p_{m,K}(f) \leq q_{m,0}(f)$ which gives the desired inclusion of topologies.

**1)** $(\mathcal{S}(\mathbb{R}^d), \tau_\mathcal{Q})$ is a locally convex t.v.s. which is also evidently Hausdorff since the family $\mathcal{Q}$ is separating. Indeed, if $q_{m,k}(f) = 0, \ \forall m, k \in \mathbb{N}_0$ then in particular $q_{0,0}(f) = \sup_{x \in \mathbb{R}^d} |f(x)| = 0$, which implies $f \equiv 0$ on $\mathbb{R}^d$.

**2)** $(\mathcal{S}(\mathbb{R}^d), \tau_\mathcal{Q})$ is a metrizable, as $\mathcal{Q}$ is countable and separating (see Proposition 1.1.5).