3) $(\mathcal{S}(\mathbb{R}^d), \tau_{\mathcal{Q}})$ is a complete. By Proposition 1.1.6, it is enough to show that it is sequentially complete. Let $(f_\nu)_{\nu\in\mathbb{N}}$ be a Cauchy sequence $\mathcal{S}(\mathbb{R}^d)$ then a fortiori we get that $(f_\nu)_{\nu\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}^\infty(\mathbb{R}^d)$ endowed with the $\mathcal{C}^\infty-$topology. Since such a space is complete, then there exists $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ s.t. $(f_\nu)_{\nu\in\mathbb{N}}$ converges to f in the the $\mathcal{C}^\infty-$topology. From this we also know that:

$$\forall \beta \in \mathbb{N}_0^d, \forall x \in \mathbb{R}^d, (D^\beta f_\nu)(x) \to (D^\beta f)(x) \text{ as } \nu \to \infty \tag{1.11}$$

We are going to prove at once that $(f_\nu)_{\nu\in\mathbb{N}}$ is converging to f in the $\tau_{\mathcal{Q}}$ topology (not only in the $\mathcal{C}^\infty-$topology) and that $f \in \mathcal{S}(\mathbb{R}^d)$.

Let $m, k \in \mathbb{N}_0$ and let $\varepsilon > 0$. As $(f_\nu)_{\nu\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}(\mathbb{R}^d)$, there exists a constant M s.t. $\forall \nu, \mu \geq M$ we have: $q_{m,k}(f_\nu - f_\mu) \leq \varepsilon$. Then fixing $\beta \in \mathbb{N}_0^d$ with $|\beta| \leq m$ and $x \in \mathbb{R}^d$ we get

$$(1 + |x|)^k \left|(D^\beta f_\nu)(x) - (D^\beta f_\mu)(x)\right| \leq \varepsilon.$$

Passing to the limit for $\mu \to \infty$ in the latter relation and using (1.11), we get

$$(1 + |x|)^k \left|(D^\beta f_\nu)(x) - (D^\beta f)(x)\right| \leq \varepsilon.$$

Hence, for all $\nu \geq M$ we have that $q_{m,k}(f_\nu - f) \leq \varepsilon$ as desired. Then by the triangular inequality it easily follows that

$$\forall m, k \in \mathbb{N}_0, q_{m,k}(f) < \infty, \text{ i.e. } f \in \mathcal{S}(\mathbb{R}^d).$$

1.3 Inductive topologies and LF-spaces

Let $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ be a family of locally convex Hausdorff t.v.s. over the field $\mathbb{K}$ of real or complex numbers (A is an arbitrary index set). Let E be a vector space over the same field $\mathbb{K}$ and, for each $\alpha \in A$, let $g_\alpha : E_\alpha \to E$ be a linear mapping. The ***inductive topology*** τ_{ind} on E w.r.t. the family $\{(E_\alpha, \tau_\alpha, g_\alpha) : \alpha \in A\}$ is the topology generated by the following basis of neighbourhoods of the origin in E:

$$\begin{aligned} \mathcal{B}_{ind} : \; = \; & \{U \subset E \, \text{convex, balanced, absorbing} : \forall \alpha \in A, \; g_\alpha^{-1}(U) \; \text{is} \\ & \text{a neighbourhood of the origin in} \, (E_\alpha, \tau_\alpha)\}. \end{aligned}$$

Then it easily follows that the space (E, τ_{ind}) is a l.c. t.v.s. (c.f. Theorem 4.1.14 in TVS-I). Note that τ_{ind} is the finest locally convex topology on E for which all the mappings g_α ($\alpha \in A$) are continuous. Suppose there exists a locally convex topology τ on E s.t. all the g_α's are continuous and $\tau_{ind} \subseteq \tau$. As (E, τ) is locally convex, there always exists a basis of neighbourhood of the origin consisting of convex, balanced, absorbing subsets of E. Then for any such a neighbourhood U of the origin in (E, τ) we have, by continuity, that $g_\alpha^{-1}(U)$ is a neighbourhood of the origin in (E_α, τ_α). Hence, $U \in \mathcal{B}_{ind}$ and so $\tau \equiv \tau_{ind}$.

It is also worth to underline that (E, τ_{ind}) is not necessarily a Hausdorff t.v.s., although all the spaces (E_α, τ_α) are Hausdorff t.v.s..

Proposition 1.3.1. *Let $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ be a family of locally convex Hausdorff t.v.s. over the field $\mathbb{K}$ and, for any $\alpha \in A$, let $g_\alpha : E_\alpha \to E$ be a linear mapping. Let E be a vector space over $\mathbb{K}$ endowed with the inductive topology τ_{ind} w.r.t. the family $\{(E_\alpha, \tau_\alpha, g_\alpha) : \alpha \in A\}$, (F, τ) an arbitrary locally convex t.v.s., and u a linear mapping from E into F. The mapping $u : E \to F$ is continuous if and only if $u \circ g_\alpha : E_\alpha \to F$ is continuous for all $\alpha \in A$.*

Proof. Suppose u is continuous and fix $\alpha \in A$. Since g_α is also continuous, we have that $u \circ g_\alpha$ is continuous as composition of continuous mappings. [3]

Conversely, suppose that for each $\alpha \in A$ the mapping $u \circ g_\alpha$ is continuous. As (F, τ) is locally convex, there always exists a basis of neighbourhoods of

[3] Alternatively: Let W be a neighbourhood of the origin in (F, τ).

Suppose u is continuous, then we have that $u^{-1}(W)$ is a neighbourhood of the origin in (E, τ_{ind}). Therefore, there exists $U \in \mathcal{B}_{ind}$ s.t. $U \subseteq u^{-1}(W)$ and so

$$g_\alpha^{-1}(U) \subseteq g_\alpha^{-1}(u^{-1}(W)) = (u \circ g_\alpha)^{-1}(U), \quad \forall \alpha \in A. \tag{1.12}$$

As by definition of $\mathcal{B}_{ind}$, each $g_\alpha^{-1}(U)$ is a neighbourhood of the origin in (E_α, τ_α), so is $(u \circ g_\alpha)^{-1}(U)$ by (1.12). Hence, all $u \circ g_\alpha$ are continuous.

the origin consisting of convex, balanced, absorbing subsets of F. Let W be such a neighbourhood. Then, by the linearity of u, we get that $u^{-1}(W)$ is a convex, balanced and absorbing subset of E. Moreover, the continuity of all $u \circ g_\alpha$ guarantees that each $(u \circ g_\alpha)^{-1}(W)$ is a neighbourhood of the origin in (E_α, τ_α), i.e. $g_\alpha^{-1}(u^{-1}(W))$ is a neighbourhood of the origin in (E_α, τ_α). Then $u^{-1}(W)$, being also convex, balanced and absorbing, must be in $\mathcal{B}_{ind}$ and so it is a neighbourhood of the origin in (E, τ_{ind}). Hence, u is continuous. $\square$

Let us consider now the case when we have a total order on the index set A and $\{E_\alpha : \alpha \in A\}$ is a family of linear subspaces of a vector space E over $\mathbb{K}$ which is directed under inclusions, i.e. $E_\alpha \subseteq E_\beta$ whenever $\alpha \leq \beta$, and s.t. $E = \cup_{\alpha \in A} E_\alpha$. For each $\alpha \in A$, let i_α be the canonical embedding of E_α in E and τ_α a topology on E_α s.t. (E_α, τ_α) is a locally convex Hausdorff t.v.s. and, whenever $\alpha \leq \beta$, the topology induced by τ_β on E_α is coarser than τ_α. The space E equipped with the inductive topology τ_{ind} w.r.t. the family $\{(E_\alpha, \tau_\alpha, i_\alpha) : \alpha \in A\}$ is said to be the ***inductive limit*** of the family of linear subspaces $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$.

An inductive limit of a family of linear subspaces $\{(E_\alpha, \tau_\alpha) : \alpha \in A\}$ is said to be a ***strict inductive limit*** if, whenever $\alpha \leq \beta$, the topology induced by τ_β on E_α coincide with τ_α.

There are even more general constructions of inductive limits of a family of locally convex t.v.s. but in the following we will focus on a more concrete family of inductive limits which are more common in applications. Namely, we are going to consider the so-called ***LF-spaces***, i.e. countable strict inductive limits of increasing sequences of Fréchet spaces. For convenience, let us explicitly write down the definition of an LF-space.

Definition 1.3.2. *Let $\{E_n : n \in \mathbb{N}\}$ be an increasing sequence of linear subspaces of a vector space E over $\mathbb{K}$, i.e. $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$, such that $E = \cup_{n \in \mathbb{N}} E_n$. For each $n \in \mathbb{N}$ let (E_n, τ_n) be a Fréchet space such that the natural embedding i_n of E_n into E_{n+1} is a topological isomorphism, i.e. the topology induced by τ_{n+1} on E_n coincides with τ_n. The space E equipped with the inductive topology τ_{ind} w.r.t. the family $\{(E_n, \tau_n, i_n) : n \in \mathbb{N}\}$ is said to be the* LF-space with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$.

A basis of neighbourhoods of the origin in the LF-space (E, τ_{ind}) with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$ is given by:
$\{U \subset E$ convex, balanced, abs. : $\forall n \in \mathbb{N}$, $U \cap E_n$ is a nbhood of o in $(E_n, \tau_n)\}$.

Note that from the construction of the LF-space (E, τ_{ind}) with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$ we know that each E_n is isomorphically embedded in the subsequent ones, but a priori we do not know if E_n is isomorphically embedded in E, i.e. if the topology induced by τ_{ind} on E_n is identical to the topology τ_n initially given on E_n. This is indeed true and it will be a consequence of the following lemma.

Lemma 1.3.3. *Let E be a locally convex space, E_0 a linear subspace of E equipped with the subspace topology, and U a convex neighbourhood of the origin in E_0. Then there exists a convex neighbourhood V of the origin in E such that $V \cap E_0 = U$.*

Proposition 1.3.4.
Let (E, τ_{ind}) be an LF-space with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$. Then

$$\tau_{ind} \restriction E_n \equiv \tau_n, \, \forall n \in \mathbb{N}.$$

From the previous proposition we can easily deduce that any LF-space is not only a locally convex t.v.s. but also Hausdorff. Indeed, if (E, τ_{ind}) is the LF-space with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$ and we denote by $\mathcal{F}(o)$ [resp. $\mathcal{F}_n(o)$] the filter of neighbourhoods of the origin in (E, τ_{ind}) [resp. in (E_n, τ_n)], then:

$$\bigcap_{V \in \mathcal{F}(o)} V = \bigcap_{V \in \mathcal{F}(o)} V \cap \left(\bigcup_{n \in \mathbb{N}} E_n \right) = \bigcup_{n \in \mathbb{N}} \bigcap_{V \in \mathcal{F}(o)} (V \cap E_n) = \bigcup_{n \in \mathbb{N}} \bigcap_{U_n \in \mathcal{F}_n(o)} U_n = \{o\},$$

which implies that (E, τ_{ind}) is Hausdorff by Corollary 2.2.4 in TVS-I.

As a particular case of Proposition 1.3.1 we get that:

Proposition 1.3.5.
Let (E, τ_{ind}) be an LF-space with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$ and (F, τ) an arbitrary locally convex t.v.s..

1. *A linear mapping u from E into F is continuous if and only if, for each $n \in \mathbb{N}$, the restriction $u \restriction E_n$ of u to E_n is continuous.*
2. *A linear form on E is continuous if and only if its restrictions to each E_n are continuous.*

Note that Propositions 1.3.4 and 1.3.5 hold for any countable strict inductive limit of an increasing sequences of locally convex Hausdorff t.v.s. (even when they are not Fréchet).