in the subsequent ones, but a priori we do not know if E_n is isomorphically embedded in E, i.e. if the topology induced by τ_{ind} on E_n is identical to the topology τ_n initially given on E_n. This is indeed true and it will be a consequence of the following lemma.

Lemma 1.3.3. *Let X be a locally convex t.v.s., X_0 a linear subspace of X equipped with the subspace topology, and U a convex neighbourhood of the origin in X_0. Then there exists a convex neighbourhood V of the origin in X such that $V \cap X_0 = U$.*

Proof.
As X_0 carries the subspace topology induced by X, there exists a neighbourhood W of the origin in X such that $U = W \cap X_0$. Since X is a locally convex t.v.s., there exists a convex neighbourhood W_0 of the origin in X such that $W_0 \subseteq W$. Let V be the convex hull of $U \cup W_0$. Then by construction we have that V is a convex neighbourhood of the origin in X and that $U \subseteq V$ which implies $U = U \cap X_0 \subseteq V \cap X_0$. We claim that actually $V \cap X_0 = U$. Indeed, let $x \in V \cap X_0$; as $x \in V$ and as U and W_0 are both convex, we may write $x = ty + (1-t)z$ with $y \in U, z \in W_0$ and $t \in [0,1]$. If $t = 1$, then $x = y \in U$ and we are done. If $0 \leq t < 1$, then $z = (1-t)^{-1}(x - ty)$ belongs to X_0 and so $z \in W_0 \cap X_0 \subseteq W \cap X_0 = U$. This implies, by the convexity of U, that $x \in U$. Hence, $V \cap X_0 \subseteq U$. □

Proposition 1.3.4.
Let (E, τ_{ind}) be an LF-space with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$. Then

$$\tau_{ind} \restriction E_n \equiv \tau_n, \ \forall n \in \mathbb{N}.$$

Proof.
($\subseteq$) Let U be a neighbourhood of the origin in (E, τ_{ind}). Then, by definition of τ_{ind}, there exists V convex, balanced and absorbing neighbourhood of the origin in (E, τ_{ind}) s.t. $V \subseteq U$ and, for each $n \in \mathbb{N}$, $V \cap E_n$ is a neighbourhood of the origin in (E_n, τ_n). Hence, $\tau_{ind} \restriction E_n \subseteq \tau_n$, $\forall n \in \mathbb{N}$.

($\supseteq$) Given $n \in \mathbb{N}$, let U_n be a convex, balanced, absorbing neighbourhood of the origin in (E_n, τ_n). Since E_n is a linear subspace of E_{n+1}, we can apply Lemma 1.3.3 (for $X = E_{n+1}$, $X_0 = E_n$ and $U = U_n$) which ensures the existence of a convex neighbourhood U_{n+1} of the origin in (E_{n+1}, τ_{n+1}) such that $U_{n+1} \cap E_n = U_n$. Then, by induction, we get that for any $k \in \mathbb{N}$ there exists a convex neighbourhood U_{n+k} of the origin in (E_{n+k}, τ_{n+k}) such that $U_{n+k} \cap E_{n+k-1} = U_{n+k-1}$. Hence, for any $k \in \mathbb{N}$, we get $U_{n+k} \cap E_n = U_n$. If we consider now $U := \bigcup_{k=1}^{\infty} U_{n+k}$, then $U \cap E_n = U_n$. Furthermore, U is a neighbourhood of the origin in (E, τ_{ind}) since $U \cap E_m$ is a neighbourhood of the origin in (E_m, τ_m) for all $m \in \mathbb{N}$. We can then conclude that $\tau_n \subseteq \tau_{ind} \restriction E_n$, $\forall n \in \mathbb{N}$. ☐

From the previous proposition we can easily deduce that any LF-space is not only a locally convex t.v.s. but also Hausdorff. Indeed, if (E, τ_{ind}) is the LF-space with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$ and we denote by $\mathcal{F}(o)$ [resp. $\mathcal{F}_n(o)$] the filter of neighbourhoods of the origin in (E, τ_{ind}) [resp. in (E_n, τ_n)], then:

$$\bigcap_{V \in \mathcal{F}(o)} V = \bigcap_{V \in \mathcal{F}(o)} V \cap \left(\bigcup_{n \in \mathbb{N}} E_n \right) = \bigcup_{n \in \mathbb{N}} \bigcap_{V \in \mathcal{F}(o)} (V \cap E_n) = \bigcup_{n \in \mathbb{N}} \bigcap_{U_n \in \mathcal{F}_n(o)} U_n = \{o\},$$

which implies that (E, τ_{ind}) is Hausdorff by Corollary 2.2.4 in TVS-I.

As a particular case of Proposition 1.3.1 we get that:

Proposition 1.3.5.
Let (E, τ_{ind}) be an LF-space with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$ and (F, τ) an arbitrary locally convex t.v.s..

1. *A linear mapping u from E into F is continuous if and only if, for each $n \in \mathbb{N}$, the restriction $u \restriction E_n$ of u to E_n is continuous.*
2. *A linear form on E is continuous if and only if its restrictions to each E_n are continuous.*

Note that Propositions 1.3.4 and 1.3.5 hold for any countable strict inductive limit of an increasing sequences of locally convex Hausdorff t.v.s. (even when they are not Fréchet).

The following results is instead typical of LF-spaces as it heavily relies on the completeness of the t.v.s. of the defining sequence.

Theorem 1.3.6. *Any LF-space is complete.*

Proof.
Let (E, τ_{ind}) be an LF-space with defining sequence $\{(E_n, \tau_n) : n \in \mathbb{N}\}$. Let $\mathcal{F}$ be a Cauchy filter on (E, τ_{ind}). Denote by $\mathcal{F}_E(o)$ the filter of neighbourhoods of the origin in (E, τ_{ind}) and consider

$$\mathcal{G} := \{A \subseteq E : \, A \supseteq M + V \text{ for some } M \in \mathcal{F}, V \in \mathcal{F}_E(o)\}.$$

1) $\mathcal{G}$ is a filter on E.
Indeed, it is clear from its definition that $\mathcal{G}$ does not contain the empty set and that any subset of E containing a set in $\mathcal{G}$ has to belong to $\mathcal{G}$. Moreover,

for any $A_1, A_2 \in \mathcal{G}$ there exist $M_1, M_2 \in \mathcal{F}$, $V_1, V_2 \in \mathcal{F}_E(o)$ s.t. $M_1 + V_1 \subseteq A_1$ and $M_2 + V_2 \subseteq A_2$; and therefore

$$A_1 \cap A_2 \supseteq (M_1 + V_1) \cap (M_2 + V_2) \supseteq (M_1 \cap M_2) + (V_1 \cap V_2).$$

The latter proves that $A_1 \cap A_2 \in \mathcal{G}$ since $\mathcal{F}$ and $\mathcal{F}_E(o)$ are both filters and so $M_1 \cap M_2 \in \mathcal{F}$ and $V_1 \cap V_2 \in \mathcal{F}_E(o)$.

2) $\mathcal{G} \subseteq \mathcal{F}$.
In fact, for any $A \in \mathcal{G}$ there exist $M \in \mathcal{F}$ and $V \in \mathcal{F}_E(o)$ s.t.

$$A \supseteq M + V \supset M + \{0\} = M$$

which implies that $A \in \mathcal{F}$ since $\mathcal{F}$ is a filter.

3) $\mathcal{G}$ is a Cauchy filter on E.
Let $U \in \mathcal{F}_E(o)$. Then there always exists $V \in \mathcal{F}_E(o)$ balanced such that $V + V - V \subseteq U$. As $\mathcal{F}$ is a Cauchy filter on (E, τ_{ind}), there exists $M \in \mathcal{F}$ such that $M - M \subseteq V$. Then

$$(M + V) - (M + V) \subseteq (M - M) + (V - V) \subseteq V + V - V \subseteq U$$

which proves that $\mathcal{G}$ is a Cauchy filter since $M + V \in \mathcal{G}$.

It is possible to show (and we do it later on) that:

$$\exists\, p \in \mathbb{N} :\ \forall A \in \mathcal{G},\ A \cap E_p \neq \emptyset \tag{1.13}$$

This property ensures that the family

$$\mathcal{G}_p := \{A \cap E_p :\ A \in \mathcal{G}\}$$

is a filter on E_p. Moreover, since $\mathcal{G}$ is a Cauchy filter on (E, τ_{ind}) and since by Proposition 1.3.4 we have $\tau_{ind} \restriction E_p = \tau_p$, $\mathcal{G}_p$ is a Cauchy filter on (E_p, τ_p). Hence, the completeness of E_p guarantees that there exists $x \in E_p$ s.t. $\mathcal{G}_p \to x$. This implies that also $\mathcal{G} \to x$ and so $\mathcal{F}_E(o) \subseteq \mathcal{G} \subseteq \mathcal{F}$ which gives $\mathcal{F} \to x$. □

Proof. of (1.13)
Suppose that (1.13) is false, i.e. $\forall n \in \mathbb{N}$, $\exists\, A_n \in \mathcal{G}$ s.t. $A_n \cap E_n = \emptyset$. By the definition of $\mathcal{G}$, this means that

$$\forall n \in \mathbb{N},\ \exists\, M_n \in \mathcal{F},\ V_n \in \mathcal{F}_E(o), \text{ s.t. } (M_n + V_n) \cap E_n = \emptyset. \tag{1.14}$$

Since E is a locally convex t.v.s., we may assume that each V_n is balanced and convex, and that $V_{n+1} \subseteq V_n$ for all $n \in \mathbb{N}$. Consider

$$W_n := conv\left(V_n \cup \bigcup_{k=1}^{n-1}(V_k \cap E_k)\right),$$

then

$$(W_n + M_n) \cap E_n = \emptyset, \forall n \in \mathbb{N}.$$

Indeed, if there exists $h \in (W_n + M_n) \cap E_n$ then $h \in E_n$ and $h \in (W_n + M_n)$. We may then write: $h = x + ty + (1-t)z$ with $x \in M_n$, $y \in V_n$, $z \in V_1 \cap E_{n-1}$ and $t \in [0,1]$. Hence, $x + ty = h - (1-t)z \in E_n$. But we also have $x + ty \in M_n + V_n$, since V_n is balanced and so $ty \in V_n$. Therefore, $x + ty \in (M_n + V_n) \cap E_n$ which contradicts (1.14).

Now let us define

$$W := conv\left(\bigcup_{k=1}^{\infty}(V_k \cap E_k)\right).$$

As W is convex and as $W \cap E_k$ contains $V_k \cap E_k$ for all $k \in \mathbb{N}$, W is a neighbourhood of the origin in (E, τ_{ind}). Moreover, as $(V_n)_{n\in\mathbb{N}}$ is decreasing, we have that for all $n \in \mathbb{N}$

$$W = conv\left(\bigcup_{k=1}^{n-1}(V_k \cap E_k) \cup \bigcup_{k=n}^{\infty}(V_k \cap E_k)\right) \subseteq conv\left(\bigcup_{k=1}^{n-1}(V_k \cap E_k) \cup V_n\right) = W_n.$$

Since $\mathcal{F}$ is a Cauchy filter on (E, τ_{ind}), there exists $B \in \mathcal{F}$ such that $B - B \subseteq W$ and so $B - B \subseteq W_n, \forall n \in \mathbb{N}$. On the other hand we have $B \cap M_n \neq \emptyset, \forall n \in \mathbb{N}$, as both B and M_n belong to $\mathcal{F}$. Hence, for all $n \in \mathbb{N}$ we get

$$B - (B \cap M_n) \subseteq B - B \subseteq W_n,$$

which implies

$$B \subseteq W_n + (B \cap M_n) \subseteq W_n + M_n$$

and so

$$B \cap E_n \subseteq (W_n + M_n) \cap E_n \overset{(1.14)}{=} \emptyset.$$

Therefore, we have got that $B \cap E_n = \emptyset$ for all $n \in \mathbb{N}$ and so that $B = \emptyset$, which is impossible as $B \in \mathcal{F}$. Hence, (1.13) must hold true. □

Example I: The space of polynomials

Let $n \in \mathbb{N}$ and $\mathbf{x} := (x_1, \dots, x_n)$. Denote by $\mathbb{R}[\mathbf{x}]$ the space of polynomials in the n variables $x_1, \dots, x_n$ with real coefficients. A canonical algebraic basis for $\mathbb{R}[\mathbf{x}]$ is given by all the monomials

$$\mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \forall \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n.$$

For any $d \in \mathbb{N}_0$, let $\mathbb{R}_d[\mathbf{x}]$ be the linear subpace of $\mathbb{R}[\mathbf{x}]$ spanned by all monomials $\mathbf{x}^\alpha$ with $|\alpha| := \sum_{i=1}^n \alpha_i \leq d$, i.e.

$$\mathbb{R}_d[\mathbf{x}] := \{f \in \mathbb{R}[\mathbf{x}] | \deg f \leq d\}.$$

Since there are exactly $\binom{n+d}{d}$ monomials $\mathbf{x}^\alpha$ with $|\alpha| \leq d$, we have that

$$dim(\mathbb{R}_d[\mathbf{x}]) = \frac{(d+n)!}{d!n!},$$

and so that $\mathbb{R}_d[\mathbf{x}]$ is a finite dimensional vector space. Hence, by Tychonoff Theorem (see Corollary 3.1.4 in TVS-I) there is a unique topology τ_e^d that makes $\mathbb{R}_d[\mathbf{x}]$ into a Hausdorff t.v.s. which is also complete and so Fréchet (as it topologically isomorphic to $\mathbb{R}^{dim(\mathbb{R}_d[x])}$ equipped with the euclidean topology).

As $\mathbb{R}[\mathbf{x}] := \bigcup_{d=0}^\infty \mathbb{R}_d[\mathbf{x}]$, we can then endow it with the inductive topology τ_{ind} w.r.t. the family of F-spaces $\left\{(\mathbb{R}_d[\mathbf{x}], \tau_e^d) : d \in \mathbb{N}_0\right\}$; thus $(\mathbb{R}[\mathbf{x}], \tau_{ind})$ is a LF-space and the following properties hold (proof as Exercise 1, Sheet 3):

a) τ_{ind} is the finest locally convex topology on $\mathbb{R}[\mathbf{x}]$,

b) every linear map f from $(\mathbb{R}[\mathbf{x}], \tau_{ind})$ into any t.v.s. is continuous.