Consider on the space of test functions $\mathcal{C}_c^\infty(\Omega)$ the projective topology τ_{proj} w.r.t. the family $\{(\mathscr{D}_t(\Omega), \tau_t, i_t) : t \in T\}$, where (for each $t \in T$) τ_t denotes the topology induced by the norm $\|\cdot\|_t$ and i_t denotes the natural embedding of $\mathcal{C}_c^\infty(\Omega)$ into $\mathscr{D}_t(\Omega)$. Then $(\mathcal{C}_c^\infty(\Omega), \tau_{proj})$ is the reduced projective limit of the family $\{(\mathscr{D}_t(\Omega), \tau_t, i_t) : t \in T\}$.

Using Sobolev embeddings theorems, it can be showed that the space of test functions $\mathcal{C}_c^\infty(\Omega)$ can be actually written as projective limit of a family of weighted Sobolev spaces which are Hilbert spaces (see Chapter I, Section 3.10 of the book [Y. M. Berezansky, Selfadjoint Operators in Spaces of Functions of Infinite Many Variables, vol. 63, Trans. Amer. Math. Soc., 1986]).

1.5 Approximation procedures in spaces of functions

When are forced to deal with "bad" functions, it is a standard strategy trying to approximate them with "nice" ones, studying the latter ones and proving that some of the properties in which we are interested, if valid for the approximating nice functions, would carry over to their limit. Usually we consider the smooth functions to be "nice" approximating functions and often (especially when we aim to compute integrals) it is convenient to look for approximating functions which also have compact support or certain growth properties at infinity. This is indeed one reason for which in this section we are going to focus on approximation by $\mathcal{C}_c^\infty$ functions.

Another reason to the usefulness of approximation techniques is that often the objects needed are extracted from t.v.s. which are spaces of functions or duals of spaces of functions. Therefore, it becomes extremely useful to understand how certain spaces of functions can be embedded in the topological duals of other spaces of functions. It is then important to know when inclusions of the kind $E' \subseteq F'$ hold (here E', F' are respectively the topological dual of the t.v.s. E and F) and what relation between E and F is connected to such an inclusion. A very much used criterion is the following one:

Proposition 1.5.1.
Given two t.v.s. (E, τ_E) and (F, τ_F). The topological dual E' of E is a linear subspace of the topological dual F' of F if:

1. *F is a linear subspace of E;*
2. *F is dense in E;*
3. *τ_F is at least as fine as the one induced by E on F, i.e. $\tau_F \supseteq (\tau_E)\restriction_F$.*

Proof.
We want to show that there exists an embedding of the vector space E' into F'. By (1) and (3), any continuous linear form on E restricted to F is a

continuous linear form on F. Moreover, if any two continuous linear forms on E define the same form on F, then they coincide on F which is by (2) a dense subset of E and, hence, they coincide everywhere in E (see TVS-I Sheet 4, Ex 3). In conclusion, we have showed that to every continuous linear form L on E corresponds one and only one continuous linear form $L\restriction_F$ on F, i.e. the map $E' \to F'$, $L \mapsto L\restriction_F$ is an embedding of vector spaces. □

Proving (1) and (3) is usually easy once we are given E and F with their respective topological structures (e.g. we know that $\mathcal{C}^\infty(\Omega) \subset \mathcal{C}^k(\Omega)$ for any integer $0 \leq k < \infty$ and that the $\mathcal{C}^\infty$-topology is finer than the $\mathcal{C}^k$-topology restricted to $\mathcal{C}^\infty(\Omega)$). Instead showing (2) can be much harder and for this we need to use approximation techniques (e.g. we will prove that $\mathcal{C}^\infty(\Omega)$ is dense in $\mathcal{C}^k(\Omega)$ for $0 \leq k < \infty$ endowed with the $\mathcal{C}^k$-topology).

Remark 1.5.2. *Remind that saying that the t.v.s. F is dense in the t.v.s. E means that every element of E is the limit of a filter on F, not necessarily of a sequence of elements in F.*

We will focus now on approximation of $\mathcal{C}^k$ functions by $\mathcal{C}^\infty$ functions with compact support. First of all, let us give an example of such a function on $\mathbb{R}^d$, which will be particularly useful in the rest of this section.

Example of a $\mathcal{C}_c^\infty$-function on $\mathbb{R}^d$

Consider for any $x \in \mathbb{R}^d$:

$$\rho(x) := \begin{cases} a\exp\left(-\frac{1}{1-|x|^2}\right) & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1 \end{cases}, \tag{1.15}$$

where

$$a := \left(\int_{\{y\in\mathbb{R}^d : |y|<1\}} \exp\left(-\frac{1}{1-|x|^2}\right)\, dx \right)^{-1}.$$

Note that

$$\int_{\mathbb{R}^d} \rho(x)\, dx = 1 \tag{1.16}$$

and $supp(\rho) := \{x \in \mathbb{R}^d : |x| \leq 1\}$ which is compact in $\mathbb{R}^d$.

Let us now check that ρ is a $\mathcal{C}^\infty$ function on $\mathbb{R}^d$. Note that the function ρ is an analytic function about every point in the open ball $\{x \in \mathbb{R}^d : |x| < 1\}$ (i.e. its Taylor's expansion about any such a point has a nonzero radius of convergence) and ρ is obviously smooth in $\{x \in \mathbb{R}^d : |x| > 1\}$, so the only question is to check what happens for $|x| = 1$. As ρ is rotation-invariant, it suffices to check if the function of one real variable:

$$\begin{cases} \exp\left(-\frac{1}{1-t^2}\right) & \text{for } |t| < 1 \\ 0 & \text{for } |t| \geq 1 \end{cases},$$

is $\mathcal{C}^\infty$ at the points $t = 1$ and $t = -1$. Since

$$\exp\left(-\frac{1}{1-t^2}\right) = \exp\left(-\frac{1}{2(1-t)}\right)\exp\left(-\frac{1}{2(1+t)}\right),$$

we actually need to only check that the function of one variable:

$$\begin{cases} \exp\left(-\frac{1}{s}\right) & \text{for } s > 0 \\ 0 & \text{for } s \leq 0 \end{cases},$$

is $\mathcal{C}^\infty$, which is a well-known fact! Hence, $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^d)$.

Let us introduce now some notations which will be useful in the following.

For any $\varepsilon > 0$, we define

$$\rho_\varepsilon(x) := \varepsilon^{-d}\rho\left(\frac{x}{\varepsilon}\right), \forall x \in \mathbb{R}^d.$$

From the properties of ρ showed above, it easily follows that $\rho_\varepsilon \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ with $supp(\rho_\varepsilon) := \{x \in \mathbb{R}^d : |x| \leq \varepsilon\}$ and that:

$$\int_{\mathbb{R}^d} \rho_\varepsilon(x)\ dx = 1. \tag{1.17}$$

Indeed, by simply using the change of variables $y = \frac{x}{\varepsilon}$ and (1.16) we get

$$\int_{\mathbb{R}^d} \rho_\varepsilon(x)\ dx = \int_{\mathbb{R}^d} \varepsilon^{-d}\rho\left(\frac{x}{\varepsilon}\right)\ dx = \int_{\mathbb{R}^d} \rho(y)\ dy = 1.$$

Given a subset S of $\mathbb{R}^d$ and a point $x \in \mathbb{R}^d$, we denote by $d(x, S)$ the Euclidean distance from x to S, i.e.

$$d(x, S) := \inf_{y \in S} |x - y|$$

and, for any $\varepsilon > 0$, we denote by $N_\varepsilon(S)$ the *neighbourhood of order* ε *of* S or $\varepsilon-$*neighbourhood of* S i.e. the set

$$N_\varepsilon(S) := \{x \in \mathbb{R}^d : d(x, S) \leq \varepsilon\}.$$

Lemma 1.5.3. *Let $f \in \mathcal{C}_c(\mathbb{R}^d)$ and for any $\varepsilon > 0$ let us define the following function on $\mathbb{R}^d$:*

$$f_\varepsilon(x) := \int_{\mathbb{R}^d} \rho_\varepsilon(x-y) f(y)\, dy.$$

Then the following hold.
a) $f_\varepsilon \in \mathcal{C}_c^\infty(\mathbb{R}^d)$.
b) The support of f_ε is contained in the neighbourhood of order ε of the support of f, i.e. $supp(f_\varepsilon) \subseteq N_\varepsilon(supp(f))$.
c) When $\varepsilon \to 0$, $f_\varepsilon \to f$ uniformly in $\mathbb{R}^d$.

Proof.
As all the derivatives w.r.t. to x of $\rho_\varepsilon(x-y)f(y)$ exist and the latter function is continuous as product of continuous functions, we can apply Leibniz' rule and differentiate f_ε w.r.t. x by passing the derivative under the integral sign. Hence, as $\rho_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^d)$, we have $f_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^d)$. Moreover, the integral expressing f_ε is actually performed over the set of points $y \in \mathbb{R}^d$ such that $y \in supp(f)$ and that $x - y \in supp(\rho_\varepsilon)$, i.e. $|x-y| \leq \varepsilon$. If $x \notin N_\varepsilon(supp(f))$ then there would not exist such points and the integral would be just zero, which means that $x \notin supp(f_\varepsilon)$. Indeed, if $x \notin N_\varepsilon(supp(f))$ then we would have for any $y \in supp(f)$ that $|x-y| \geq d(x, supp(f)) > \varepsilon$, i.e. $x - y \notin supp(\rho_\varepsilon)$, which gives $f_\varepsilon(x) = 0$ and so (2). The latter also guarantees that f_ε has compact support and so we can conclude that $f_\varepsilon \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, i.e. (1) holds.

It remains to show that (3) holds.

As f is a continuous function which is identically zero outside a compact set, f is uniformly continuous on $\mathbb{R}^d$, i.e. $\forall\, \eta > 0$, $\exists\, \overline{\varepsilon} > 0$ s.t. $\forall x, y \in \mathbb{R}^d$

$$|x-y| < \overline{\varepsilon} \ \text{ implies } \ |f(x) - f(y)| \leq \eta. \tag{1.18}$$

Moreover, for any $\varepsilon > 0$ and any $x \in \mathbb{R}^d$, by using (1.17) we easily get that:

$$\int_{\mathbb{R}^d} \rho_\varepsilon(x-y)dy = \int_{\mathbb{R}^d} \rho_\varepsilon(-z)dz = \int_{\mathbb{R}^d} \rho_\varepsilon(z)dz = 1. \tag{1.19}$$

Therefore, for all $x \in \mathbb{R}^d$ we can write:

$$f(x) - f_\varepsilon(x) = \int_{\mathbb{R}^d} \rho_\varepsilon(x-y)(f(x) - f(y))dy$$

which together with (1.19) gives that:

$$|f(x)-f_\varepsilon(x)| \leq \left(\sup_{\substack{y\in\mathbb{R}^d \\ |x-y|<\varepsilon}} |f(x) - f(y)| \right) \int_{\mathbb{R}^d} \rho_\varepsilon(x-y)dy \leq \sup_{\substack{y\in\mathbb{R}^d \\ |x-y|<\varepsilon}} |f(x)-f(y)|.$$

Hence, using the latter together with (1.18), we get that $\forall \eta > 0$, $\exists \bar{\varepsilon} > 0$ s.t. $\forall x \in \mathbb{R}^d, \forall \varepsilon \leq \bar{\varepsilon}$

$$|f(x) - f_\varepsilon(x)| \leq \sup_{\substack{y \in \mathbb{R}^d \\ |x-y|<\varepsilon}} |f(x) - f(y)| \leq \sup_{\substack{y \in \mathbb{R}^d \\ |x-y|<\bar{\varepsilon}}} |f(x) - f(y)| \leq \eta,$$

i.e. $f_\varepsilon \to f$ uniformly on $\mathbb{R}^d$ when $\varepsilon \to 0$. □

Corollary 1.5.4. *Let $f \in \mathcal{C}_c^k(\mathbb{R}^d)$ with $0 \leq k \leq \infty$ integer and for any $\varepsilon > 0$ let us define f_ε as in Lemma 1.5.3. Then, for any $p = (p_1, \ldots, p_d) \in \mathbb{N}_0^d$ such that $|p| \leq k$, $D^p f_\varepsilon \to D^p f$ uniformly on $\mathbb{R}^d$ when $\varepsilon \to 0$.*

Proof. (Christmas assignment) □

Before proving our approximation theorem by $\mathcal{C}_c^\infty$ functions, let us recall that a sequence of subsets S_j of $\mathbb{R}^d$ converges to a subset S of $\mathbb{R}^d$ if:

$$\forall \varepsilon > 0, \exists J_\varepsilon > 0 \text{ s.t. } \forall j \geq J_\varepsilon, S_j \subset N_\varepsilon(S) \text{ and } S \subset N_\varepsilon(S_j).$$

Theorem 1.5.5. *Let $0 \leq k \leq \infty$ be an integer and Ω be an open set of $\mathbb{R}^d$. Any function $f \in \mathcal{C}^k(\Omega)$ is the limit of a sequence $(f_j)_{j \in \mathbb{N}}$ of functions in $\mathcal{C}_c^\infty(\Omega)$ such that, for each compact subset K of Ω, the set $K \cap supp(f_j)$ converges to $K \cap supp(f)$.*

Proof.
Let $(\Omega_j)_{j \in \mathbb{N}_0}$ be a sequence of open subsets whose union is equal to Ω and such that, for each $j \geq 1$, $\overline{\Omega_{j-1}}$ is compact and contained in Ω_j. Define $d_j := d(\overline{\Omega_{j-1}}, \Omega_j^c)$, where Ω_j^c denotes the complement of Ω_j, then we have $d_j > 0$ for all $j \in \mathbb{N}$. We can therefore construct for each $j \in \mathbb{N}$ a function $g_j \in \mathcal{C}(\Omega)$ with the following properties:

$$g_j(x) = 1 \text{ if } d(x, \Omega_j^c) \geq \frac{3}{4} d_j, \text{ and } g_j(x) = 0 \text{ if } d(x, \Omega_j^c) \leq \frac{d_j}{2}.$$

Note that $supp(g_j) \subseteq \overline{\Omega_j}$ and so $g_j \in \mathcal{C}_c(\Omega)$. Define $\varepsilon_j := \frac{d_j}{4}$ and consider the function:

$$h_j(x) := \int_{\mathbb{R}^d} \rho_{\varepsilon_j}(x - y) g_j(y)\, dy.$$

If $x \in \Omega_{j-1}$ and $x - y \in supp(\rho_{\varepsilon_j})$, i.e. $|x - y| \leq \frac{d_j}{4}$, then we have:

$$d(y, \Omega_j^c) \geq d(x, \Omega_j^c) - |x - y| \geq d_j - \frac{d_j}{4} = \frac{3}{4} d_j$$

which implies $g_j(y) = 1$ and so $h_j(x) = \int_{\mathbb{R}^d} \rho_{\varepsilon_j}(x-y)\, dy = 1$ in view of (1.19). Hence, $h_j \equiv 1$ on Ω_{j-1}.

Since $g_j \in \mathcal{C}_c(\Omega)$, we can apply Lemma 1.5.3 to the functions h_j and get that $h_j \in \mathcal{C}_c^\infty(\Omega)$. Moreover, as $h_j \equiv 1$ on Ω_{j-1}, it is clear that $h_j \to 1$ in $\mathcal{C}^\infty(\Omega)$ when $j \to \infty$.

Given any function $f \in \mathcal{C}^k(\Omega)$, we have that $h_j f \in \mathcal{C}_c^k(\Omega)$ as it is product of a $\mathcal{C}^\infty$ function with a $\mathcal{C}^k$ function and $supp(h_j f) \subseteq supp(h_j) \cap supp(f) \subseteq supp(h_j)$ which is compact. Also, since $h_j \to 1$ in $\mathcal{C}^\infty(\Omega)$ as $j \to \infty$, we have that $h_j f \to f$ in $\mathcal{C}^k(\Omega)$ as $j \to \infty$.