Hence, using the latter together with (1.18), we get that $\forall\,\eta > 0$, $\exists\,\overline{\varepsilon} > 0$ s.t. $\forall x \in \mathbb{R}^d, \forall \varepsilon \leq \overline{\varepsilon}$

$$|f(x) - f_\varepsilon(x)| \leq \sup_{\substack{y \in \mathbb{R}^d \\ |x-y|<\varepsilon}} |f(x) - f(y)| \leq \sup_{\substack{y \in \mathbb{R}^d \\ |x-y|<\overline{\varepsilon}}} |f(x) - f(y)| \leq \eta,$$

i.e. $f_\varepsilon \to f$ uniformly on $\mathbb{R}^d$ when $\varepsilon \to 0$. □

Corollary 1.5.4. *Let $f \in \mathcal{C}_c^k(\mathbb{R}^d)$ with $0 \leq k \leq \infty$ integer and for any $\varepsilon > 0$ let us define f_ε as in Lemma 1.5.3. Then, for any $p = (p_1, \ldots, p_d) \in \mathbb{N}_0^d$ such that $|p| \leq k$, $D^p f_\varepsilon \to D^p f$ uniformly on $\mathbb{R}^d$ when $\varepsilon \to 0$.*

Proof. (Christmas assignment) □

Before proving our approximation theorem by $\mathcal{C}_c^\infty$ functions, let us recall that a sequence of subsets S_j of $\mathbb{R}^d$ converges to a subset S of $\mathbb{R}^d$ if:

$$\forall \varepsilon > 0,\ \exists J_\varepsilon > 0 \text{ s.t. } \forall j \geq J_\varepsilon,\ S_j \subset N_\varepsilon(S) \text{ and } S \subset N_\varepsilon(S_j).$$

Theorem 1.5.5. *Let $0 \leq k \leq \infty$ be an integer and Ω be an open set of $\mathbb{R}^d$. Any function $f \in \mathcal{C}^k(\Omega)$ is the limit of a sequence $(f_j)_{j\in\mathbb{N}}$ of functions in $\mathcal{C}_c^\infty(\Omega)$ such that, for each compact subset K of Ω, the set $K \cap supp(f_j)$ converges to $K \cap supp(f)$.*

Proof.
Let $(\Omega_j)_{j\in\mathbb{N}_0}$ be a sequence of open subsets whose union is equal to Ω and such that, for each $j \geq 1$, $\overline{\Omega_{j-1}}$ is compact and contained in Ω_j. Define $d_j := d(\overline{\Omega_{j-1}}, \Omega_j^c)$, where Ω_j^c denotes the complement of Ω_j, then we have $d_j > 0$ for all $j \in \mathbb{N}$. We can therefore construct for each $j \in \mathbb{N}$ a function $g_j \in \mathcal{C}(\Omega)$ with the following properties:

$$g_j(x) = 1 \text{ if } d(x, \Omega_j^c) \geq \frac{3}{4} d_j, \text{ and } g_j(x) = 0 \text{ if } d(x, \Omega_j^c) \leq \frac{d_j}{2}.$$

Note that $supp(g_j) \subseteq \overline{\Omega_j}$ and so $g_j \in \mathcal{C}_c(\Omega)$. Define $\varepsilon_j := \frac{d_j}{4}$ and consider the function:

$$h_j(x) := \int_{\mathbb{R}^d} \rho_{\varepsilon_j}(x - y) g_j(y)\, dy.$$

If $x \in \Omega_{j-1}$ and $x - y \in supp(\rho_{\varepsilon_j})$, i.e. $|x - y| \leq \frac{d_j}{4}$, then we have:

$$d(y, \Omega_j^c) \geq d(x, \Omega_j^c) - |x - y| \geq d_j - \frac{d_j}{4} = \frac{3}{4} d_j$$

which implies $g_j(y) = 1$ and so $h_j(x) = \int_{\mathbb{R}^d} \rho_{\varepsilon_j}(x-y)\, dy = 1$ in view of (1.19). Hence, $h_j \equiv 1$ on Ω_{j-1}.

Since $g_j \in \mathcal{C}_c(\Omega)$, we can apply Lemma 1.5.3 to the functions h_j and get that $h_j \in \mathcal{C}_c^\infty(\Omega)$. Moreover, as $h_j \equiv 1$ on Ω_{j-1}, it is clear that $h_j \to 1$ in $\mathcal{C}^\infty(\Omega)$ when $j \to \infty$.

Given any function $f \in \mathcal{C}^k(\Omega)$, we have that $h_j f \in \mathcal{C}_c^k(\Omega)$ as it is product of a $\mathcal{C}^\infty$ function with a $\mathcal{C}^k$ function and $supp(h_j f) \subseteq supp(h_j) \cap supp(f) \subseteq supp(h_j)$ which is compact. Also, since $h_j \to 1$ in $\mathcal{C}^\infty(\Omega)$ as $j \to \infty$, we have that $h_j f \to f$ in $\mathcal{C}^k(\Omega)$ as $j \to \infty$.

Note that if K is an arbitrary compact subset of Ω, then there exists $j \in \mathbb{N}$ large enough that $K \subset \Omega_{j-1}$ and so s.t. $h_j(x) = 1$ for all $x \in K$, which implies

$$supp(h_j f) \cap K = supp(f) \cap K. \tag{1.20}$$

So far we have approximated $f \in \mathcal{C}^k(\Omega)$ by functions in $\mathcal{C}_c^k(\Omega)$, namely the functions $h_j f$, but we want to approximate f by functions $\mathcal{C}_c^\infty(\Omega)$.

Suppose that $0 \leq k < \infty$. By applying Lemma 1.5.3 and Corollary 1.5.4 to each $h_j f \in \mathcal{C}_c^k(\Omega)$ we can construct a function $f_j \in \mathcal{C}_c^\infty(\Omega)$ such that $supp(f_j) \subseteq N_{\frac{1}{j}}(supp(h_j f))$ and for any $p = (p_1, \ldots, p_d) \in \mathbb{N}_0^d$ with $|p| \leq k$ we have that

$$\exists j_1^p \in \mathbb{N} : \forall j \geq j_1^p, \sup_{x \in \Omega} |D^p (f_j(x) - h_j(x) f(x))| \leq \frac{1}{j}.$$

Hence, we have

$$\exists j_1 \in \mathbb{N} : \forall j \geq j_1, \sup_{|p| \leq k} \sup_{x \in \Omega} |D^p (f_j(x) - h_j(x) f(x))| \leq \frac{1}{j}.$$

As we also know that $h_j f \to f$ as $j \to \infty$ in the $\mathcal{C}^k-$topology, it is easy to see that $f_j \to f$ as $j \to \infty$ in the $\mathcal{C}^k$-topology.

Let K be a compact subset of Ω, then there exists $\tilde{j} \in \mathbb{N}$ large enough that $K \subset \Omega_{\tilde{j}-1}$. Hence, for any $j \geq \tilde{j}$ we have that (1.20) holds and also that $supp(f_j) \subseteq N_{\frac{1}{j}}(supp(h_j f))$. These properties jointly imply that

$$K \cap supp(f_j) \subseteq N_{\frac{1}{j}}(K \cap supp(h_j f)) = N_{\frac{1}{j}}(K \cap supp(f)), \quad \forall j \geq \tilde{j}.$$

Therefore, for any $\varepsilon > 0$ we can take $J_\varepsilon^{(1)} := \max\{\tilde{j}, \frac{1}{\varepsilon}\}$ and so for any $j \geq J_\varepsilon^{(1)}$ we get $K \cap supp(f_j) \subseteq N_\varepsilon(K \cap supp(f))$.

Also for any $\varepsilon > 0$ there exists $c > 0$ such that

$$K \cap supp(f) \subseteq \{x \in K : |f(x)| \geq c\} + \{x \in \Omega : |x| \leq \varepsilon\}. \tag{1.21}$$

If we choose now $J_\varepsilon^{(2)} \in \mathbb{N}$ large enough that both $K \subset \Omega_{J_\varepsilon^{(2)}-1}$ and $\frac{1}{J_\varepsilon^{(2)}} \leq \frac{c}{2}$, then (by the uniform convergence of f_j to f) for any $x \in K$ and any $j \geq J_\varepsilon^{(2)}$ we have that $|f_j(x) - f(x)| \leq \frac{1}{j} \leq \frac{c}{2}$ and so that

$$\{x \in K : |f(x)| \geq c\} \subseteq K \cap supp(f_j). \tag{1.22}$$

Indeed, if for any $x \in K$ such that $|f(x)| \geq c$ we had $f_j(x) = 0$, then we would get $c \leq |f(x)| = |f_j(x) - f(x)| \leq \frac{c}{2}$ which is a contradiction.

Then, by (1.21) and (1.22), we have that:

$$K \cap supp(f) \subseteq (K \cap supp(f_j)) + \{x \in \Omega : |x| \leq \varepsilon\} =: A_j$$

and it is easy to show that A_j is actually contained in $N_\varepsilon(K \cap supp(f_j))$. In fact, if $x \in A_j$ then $x = z + w$ for some $z \in K \cap supp(f_j)$ and $w \in \Omega$ s.t. $|w| \leq \varepsilon$; thus we have

$$d(x, K \cap supp(f_j)) = \inf_{y \in K \cap supp(f_j)} |z + w - y| \leq \inf_{y \in K \cap supp(f_j)} |z - y| + |w| = |w| \leq \varepsilon.$$

Hence, for all $j \geq \max\{J_\varepsilon^{(1)}, J_\varepsilon^{(2)}\}$ we have both $K \cap supp(f_j) \subseteq N_\varepsilon(K \cap supp(f))$ and $K \cap supp(f) \subseteq N_\varepsilon(K \cap supp(f_j))$.

It is easy to work out the analogous proof in the case when $k = \infty$ (do it as an additional exercise). □

We therefore have the following two corollaries.

Corollary 1.5.6. *Let $0 \leq k \leq \infty$ be an integer and Ω be an open set of $\mathbb{R}^d$. $\mathcal{C}_c^\infty(\Omega)$ is sequentially dense in $\mathcal{C}^k(\Omega)$.*

Corollary 1.5.7. *Let $0 \leq k \leq \infty$ be an integer and Ω be an open set of $\mathbb{R}^d$. $\mathcal{C}_c^\infty(\Omega)$ is dense in $\mathcal{C}^k(\Omega)$.*

With a quite similar proof scheme to the one used in Theorem 1.5.5 (for all the details see the first part of [2, Chapter 15]) is possible to show that:

Proposition 1.5.8. *Let $0 \leq k \leq \infty$ be an integer and Ω be an open set of $\mathbb{R}^d$. Every function in $\mathcal{C}_c^k(\Omega)$ is the limit in the $\mathcal{C}^k$-topology of a sequence of polynomials in $d-$variables.*

Hence, by combining this result with Corollary 1.5.6, we get that

Corollary 1.5.9. *Let $0 \leq k \leq \infty$ be an integer and Ω be an open set of $\mathbb{R}^d$. Polynomials with d variables in Ω form a sequentially dense linear subspace of $\mathcal{C}^k(\Omega)$.*

Chapter 2

Bounded subsets of topological vector spaces

In this chapter we will study the notion of bounded set in any t.v.s. and analyzing some properties which will be useful in the following and especially in relation with duality theory. Since compactness plays an important role in the theory of bounded sets, we will start this chapter by recalling some basic definitions and properties of compact subsets of a t.v.s..

2.1 Preliminaries on compactness

Let us recall some basic definitions of compact subset of a topological space (not necessarily a t.v.s.)

Definition 2.1.1. *A topological space X is said to be* compact *if X is Hausdorff and if every open covering $\{\Omega_i\}_{i\in I}$ of X contains a finite subcovering, i.e. for any collection $\{\Omega_i\}_{i\in I}$ of open subsets of X s.t. $\bigcup_{i\in I}\Omega_i = X$ there exists a finite subset $J\subseteq I$ s.t. $\bigcup_{j\in J}\Omega_j = X$.*

By going to the complements, we obtain the following equivalent definition of compactness.

Definition 2.1.2. *A topological space X is said to be* compact *if X is Hausdorff and if every family of closed sets $\{F_i\}_{i\in I}$ whose intersection is empty contains a finite subfamily whose intersection is empty, i.e. for any collection $\{F_i\}_{i\in I}$ of closed subsets of X s.t. $\bigcap_{i\in I}F_i = \emptyset$ there exists a finite subset $J\subseteq I$ s.t. $\bigcap_{j\in J}F_j = \emptyset$.*

Definition 2.1.3. *A subset K of a topological space X is said to be* compact *if K endowed with the topology induced by X is Hausdorff and for any collection $\{\Omega_i\}_{i\in I}$ of open subsets of X s.t. $\bigcup_{i\in I}\Omega_i \supseteq K$ there exists a finite subset $J\subseteq I$ s.t. $\bigcup_{j\in J}\Omega_j \supseteq K$.*

Let us state without proof a few well-known properties of compact spaces.

Proposition 2.1.4.

1. *A closed subset of a compact space is compact.*
2. *Finite unions of compact sets are compact.*
3. *Arbitrary intersections of compact subsets of a Hausdorff topological space are compact.*
4. *Let f be a continuous mapping of a compact space X into a Hausdorff topological space Y. Then $f(X)$ is a compact subset of Y.*
5. *Let f be a one-to-one-continuous mapping of a compact space X onto a compact space Y. Then f is a homeomorphism.*
6. *Let τ_1, τ_2 be two Hausdorff topologies on a set X. If $\tau_1 \subseteq \tau_2$ and (X, τ_2) is compact then $\tau_1 \equiv \tau_2$.*

In the following we will almost always be concerned with compact subsets of a Hausdorff t.v.s. E carrying the topology induced by E, and so which are themselves Hausdorff t.v.s.. Therefore, we are now introducing a useful characterization of compactness in Hausdorff topological spaces.

Theorem 2.1.5. *Let X be a Hausdorff topological space. X is compact if and only if every filter on X has at least one accumulation point.*

Proof.
Suppose that X is compact. Let $\mathcal{F}$ be a filter on X and $\mathcal{C} := \{\overline{M} : M \in \mathcal{F}\}$. As $\mathcal{F}$ is a filter, no finite intersection of elements in $\mathcal{C}$ can be empty. Therefore, by compactness, the intersection of all elements in $\mathcal{C}$ cannot be empty. Then there exists at least a point $x \in \overline{M}$ for all $M \in \mathcal{F}$, i.e. x is an accumulation point of $\mathcal{F}$. Conversely, suppose that every filter on X has at least one accumulation point. Let ϕ be a family of closed sets whose total intersection is empty. To show that X is compact, we need to show that there exists a finite subfamily of ϕ whose intersection is empty. Suppose by contradiction that no finite subfamily of ϕ has empty intersection. Then the family ϕ' of all the finite intersections of subsets belonging to ϕ forms a basis of a filter $\mathcal{F}$ on X. By our initial assumption, $\mathcal{F}$ has an accumulation point, say x. Thus, x belongs to the closure of any subset belonging to $\mathcal{F}$ and in particular to any set belonging to ϕ' (as the elements in ϕ' are themselves closed). This means that x belongs to the intersection of all the sets belonging to ϕ' , which is the same as the intersection of all the sets belonging to ϕ. But we had assumed the latter to be empty and so we have a contradiction. □

Corollary 2.1.6. *A compact subset K of a Hausdorff topological space X is closed.*

Proof.
Let K be a compact subset of a Hausdorff topological space X and let $x \in \overline{K}$. Denote by $\mathcal{F}(x) \restriction K$ the filter generated by all the sets $U \cap K$ where $U \in \mathcal{F}(x)$ (i.e. U is a neighbourhood of x in X). By Theorem 2.1.5, $\mathcal{F}(x) \restriction K$ has an accumulation point $x_1 \in K$. We claim that $x_1 \equiv x$, which implies $\overline{K} = K$ and so K closed. In fact, if $x_1 \neq x$ then there would exist $U \in \mathcal{F}(x)$ s.t. $X \setminus U$ is a neighbourhood of x_1 and thus $x_1 \neq \overline{U \cap K}$, which would contradict the fact that x_1 is an accumulation point $\mathcal{F}(x) \restriction K$. □

Last but not least let us recall the following two definitions.

Definition 2.1.7. *A subset A of a topological space X is said to be* relatively compact *if the closure $\overline{A}$ of A is compact in X.*

Definition 2.1.8. *A subset A of a Hausdorff t.v.s. E is said to be precompact if A is relatively compact when viewed as a subset of the completion $\hat{E}$ of E.*

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