*Proof.*
Let $K$ be a compact subset of a Hausdorff topological space $X$ and let $x \in \overline{K}$. Denote by $\mathcal{F}(x) \restriction K$ the filter generated by all the sets $U \cap K$ where $U \in \mathcal{F}(x)$ (i.e. $U$ is a neighbourhood of $x$ in $X$). By Theorem 2.1.5, $\mathcal{F}(x) \restriction K$ has an accumulation point $x_1 \in K$. We claim that $x_1 \equiv x$, which implies $\overline{K} = K$ and so $K$ closed. In fact, if $x_1 \neq x$ then there would exist $U \in \mathcal{F}(x)$ s.t. $X \setminus U$ is a neighbourhood of $x_1$ and thus $x_1 \neq \overline{U \cap K}$, which would contradict the fact that $x_1$ is an accumulation point $\mathcal{F}(x) \restriction K$. □

Last but not least let us recall the following two definitions.

**Definition 2.1.7.** *A subset $A$ of a topological space $X$ is said to be* relatively compact *if the closure $\overline{A}$ of $A$ is compact in $X$.*

**Definition 2.1.8.** *A subset $A$ of a Hausdorff t.v.s. $E$ is said to be precompact if $A$ is relatively compact when viewed as a subset of the completion $\hat{E}$ of $E$.*

## 2.2 Bounded subsets: definition and general properties

**Definition 2.2.1.** *A subset $B$ of a t.v.s. $E$ is said to be* bounded *if for every $U$ neighbourhood of the origin in $E$ there exists $\lambda > 0$ such that $B \subseteq \lambda U$.*

In rough words this means that a subset $B$ of $E$ is bounded if $B$ can be swallowed by any neighbourhood of the origin.

**Proposition 2.2.2.**

1. *If any element in some basis of neighbourhoods of the origin of a t.v.s. swallows a subset, then such a subset is bounded.*
2. *The closure of a bounded set is bounded.*
3. *Finite unions of bounded sets are bounded sets.*
4. *Any subset of a bounded set is a bounded set.*

*Proof.* Let $E$ be a t.v.s. and $B \subset E$.

1. Suppose that $\mathcal{N}$ is a basis of neighbourhoods of the origin $o$ in $E$ such that for every $N \in \mathcal{N}$ there exists $\lambda_N > 0$ with $B \subseteq \lambda_N N$. Then, by definition of basis of neighbourhoods of $o$, for every $U$ neighbourhood of $o$ in $E$ there exists $M \in \mathcal{N}$ s.t. $M \subseteq U$. Hence, there exists $\lambda_M > 0$ s.t. $B \subseteq \lambda_M M \subseteq \lambda U$, i.e. $B$ is bounded.
2. Suppose that $B$ is bounded in $E$. Then, as there always exists a basis $\mathcal{C}$ of neighbourhoods of the origin in $E$ consisting of closed sets (see Corollary 2.1.14-a) in TVS-I), we have that for any $C \in \mathcal{C}$ there exists $\lambda > 0$ s.t.

$B \subseteq \lambda C$ and thus $\overline{B} \subseteq \overline{\lambda C} = \lambda\overline{C} = \lambda C$. By Proposition 2.2.2-1, this is enough to conclude that $\overline{B}$ is bounded in $E$.

3. Let $n \in \mathbb{N}$ and $B_1, \dots, B_n$ bounded subsets of $E$. As there always exists a basis $\mathcal{B}$ of balanced neighbourhoods of the origin in $E$ (see Corollary 2.1.14-b) in TVS-I), we have that for any $V \in \mathcal{B}$ there exist $\lambda_1, \dots, \lambda_n > 0$ s.t. $B_i \subseteq \lambda_i V$ for all $i = 1, \dots, n$. Then $\bigcup_{i=1}^n B_i \subseteq \bigcup_{i=1}^n \lambda_i V \subseteq \left(\max_{i=1,\dots,n} \lambda_i\right) V$, which implies the boundedness of $\bigcup_{i=1}^n B_i$ by Proposition 2.2.2-1.
4. Let $B$ be bounded in $E$ and let $A$ be a subset of $B$. The boundedness of $B$ guarantees that for any neighbourhood $U$ of the origin in $E$ there exists $\lambda > 0$ s.t. $\lambda U$ contains $B$ and so $A$. Hence, $A$ is bounded.

□

The properties in Proposition 2.2.2 lead to the following definition which is dually corresponding to the notion of basis of neighbourhoods.

**Definition 2.2.3.** *Let $E$ be a t.v.s. A family $\{B_\alpha\}_{\alpha \in I}$ of bounded subsets of $E$ is called a* basis of bounded subsets *of $E$ if for every bounded subset $B$ of $E$ there is $\alpha \in I$ s.t. $B \subseteq B_\alpha$.*

This duality between neighbourhoods and bounded subsets will play an important role in the study of the strong topology on the dual of a t.v.s.

Which sets do we know to be bounded in any t.v.s.?

- Singletons are bounded in any t.v.s., as every neighbourhood of the origin is absorbing.
- Finite subsets in any t.v.s. are bounded as finite union of singletons.

**Proposition 2.2.4.** *Compact subsets of a t.v.s. are bounded.*

*Proof.* Let $E$ be a t.v.s. and $K$ be a compact subset of $E$. For any neighbourhood $U$ of the origin in $E$ we can always find an open and balanced neighbourhood $V$ of the origin s.t. $V \subseteq U$. Then we have

$$K \subseteq E = \bigcup_{n=0}^{\infty} nV.$$

From the compactness of $K$, it follows that there exist finitely many integers $n_1, \dots, n_r \in \mathbb{N}_0$ s.t.

$$K \subseteq \bigcup_{i=1}^{r} n_i V \subseteq \left(\max_{i=1,\dots,r} n_i\right) V \subseteq \left(\max_{i=1,\dots,r} n_i\right) U.$$

Hence, $K$ is bounded in $E$. □

This together with Corollary 2.1.6 gives that in any Hausdorff t.v.s. a compact subset is always bounded and closed. In finite dimensional Hausdorff t.v.s. we know that also the converse holds (because of Theorem 3.1.1 in TVS-I) and thus the ***Heine-Borel property*** always holds, i.e.

$$K \text{ compact} \Leftrightarrow K \text{ bounded and closed.}$$

This is not true, in general, in infinite dimensional t.v.s.

**Example 2.2.5.**
*Let $E$ be an infinite dimensional normed space. If every bounded and closed subset in $E$ were compact, then in particular all the balls centered at the origin would be compact. Then the space $E$ would be locally compact and so finite dimensional as proved in Theorem 3.2.1 in TVS-I, which gives a contradiction.*

There is however an important class of infinite dimensional t.v.s., the so-called *Montel spaces*, in which the Heine-Borel property holds. Note that $\mathcal{C}^\infty(\mathbb{R}^d), \mathcal{C}_c^\infty(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d)$ are all Montel spaces.

Proposition 2.2.4 provides some further interesting classes of bounded subsets in a Hausdorff t.v.s..

**Corollary 2.2.6.** *Precompact subsets of a Hausdorff t.v.s. are bounded.*

*Proof.*
Let $K$ be a precompact subset of $E$. By Definition 2.1.8, this means that the closure $\hat{K}$ of $K$ in in the completion $\hat{E}$ of $E$ is compact. Let $U$ be any neighbourhood of the origin in $E$. Since the injection $E \to \hat{E}$ is a topological monomorphism, there is a neighbourhood $\hat{U}$ of the origin in $\hat{E}$ such that $U = \hat{U} \cap E$. Then, by Proposition 2.2.4, there is a number $\lambda > 0$ such that $\hat{K} \subseteq \lambda\hat{U}$. Hence, we get

$$K \subseteq \hat{K} \cap E \subseteq \lambda\hat{U} \cap E = \lambda\hat{U} \cap \lambda E = \lambda(\hat{U} \cap E) = \lambda U. \qquad \square$$

**Corollary 2.2.7.** *Let $E$ be a Hausdorff t.v.s. The union of a converging sequence in $E$ and of its limit is a compact and so bounded closed subset in $E$.*

*Proof.* (Christmas assignment) $\square$

**Corollary 2.2.8.** *Let $E$ be a Hausdorff t.v.s. Any Cauchy sequence in $E$ is bounded.*

*Proof.* By using Corollary 2.2.7, one can show that any Cauchy sequence $S$ in $E$ is a precompact subset of $E$. Then it follows by Corollary 2.2.6 that $S$ is bounded in $E$. $\square$

Note that a Cauchy sequence $S$ in a Hausdorff t.v.s. $E$ is not necessarily relatively compact in $E$. Indeed, if this were the case, then its closure in $E$ would be compact and so, by Theorem 2.1.5, the filter associated to $S$ would have an accumulation point $x \in E$. Hence, by Proposition 1.3.8 and Proposition Proposition 1.1.30 in TVS-I, we get $S \to x \in E$ which is not necessarily true unless $E$ is complete.

**Proposition 2.2.9.** *The image of a bounded set under a continuous linear map between t.v.s. is a bounded set.*

*Proof.* Let $E$ and $F$ be two t.v.s. ,$f : E \to F$ be linear and continuous, and $B \subseteq E$ be bounded. Then for any neighbourhood $V$ of the origin in $F$, $f^{-1}(V)$ is a neighbourhood of the origin in $E$. By the boundedness of $B$ in $E$, if follows that there exists $\lambda > 0$ s.t. $B \subseteq \lambda f^{-1}(V)$ and thus, $f(B) \subseteq \lambda V$. Hence, $f(B)$ is a bounded subset of $F$. □

**Corollary 2.2.10.** *Let $L$ be a continuous linear functional on a t.v.s. $E$. If $B$ is a bounded subset of $E$, then $\sup_{x \in B} |L(x)| < \infty$.*

Let us now introduce a general characterization of bounded sets in terms of sequences.

**Proposition 2.2.11.** *Let $E$ be any t.v.s.. A subset $B$ of $E$ is bounded if and only if every sequence contained in $B$ is bounded in $E$.*

*Proof.* The necessity of the condition is obvious from Proposition 2.2.2-4. Let us prove its sufficiency. Suppose that $B$ is unbounded and let us show that it contains a sequence of points which is also unbounded. As $B$ is unbounded, there exists a neighbourhood $U$ of the origin in $E$ s.t. for all $\lambda > 0$ we have $B \not\subseteq \lambda U$. W.l.o.g. we can assume $U$ balanced. Then

$$\forall\, n \in \mathbb{N}, \exists\, x_n \in B \text{ s.t. } x_n \notin nU. \tag{2.1}$$

The sequence $\{x_n\}_{n \in \mathbb{N}}$ cannot be bounded. In fact, if it was bounded then there would exist $\mu > 0$ s.t. $\{x_n\}_{n \in \mathbb{N}} \subseteq \mu U \subseteq mU$ for some $m \in \mathbb{N}$ with $m \geq \mu$ and in particular $x_m \in mU$, which contradicts (2.1). □