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TOPOLOGICAL VECTOR SPACES II–WS 2017/2018

Christmas Assignment

This exercise sheet aims to assess your progress and to explicitly work out more details of some of the results proposed in the previous lectures. Please, hand in your solutions in postbox 13 near F411 by Monday the 8th of January at noon. The solutions to this assignment will be discussed in the tutorial on Wednesday the 10th of January (13:30–15:00) in D404.

1) Let Ω be an open subset of $\mathbb{R}^d$ and consider the two constructions of $C_c^\infty(\Omega)$ as inductive and projective limit given in Section 1.3–Example II and in Section 1.4–Example II, respectively. Show that the projective topology defined on $C_c^\infty(\Omega)$ is coarser than the inductive topology defined on the same vector space.

2) Let $f \in \mathcal{C}_c^k(\mathbb{R}^d)$ with an integer $0 \leq k \leq \infty$ and for any $\varepsilon > 0$ let us define the following function on $\mathbb{R}^d$:

$$f_\varepsilon(x) := \int_{\mathbb{R}^d} \rho_\varepsilon(x-y) f(y)\, \mathrm{d}y.$$

Prove that, for any $p = (p_1, \ldots, p_d) \in \mathbb{N}_0^d$ such that $|p| \leq k$, $D^p f_\varepsilon \to Df$ uniformly on $\mathbb{R}^d$ when $\varepsilon \to 0$ (this corresponds to Corollary 1.5.4 in the lecture notes).

3) Show that if $1 \leq p < \infty$, then the space $\mathcal{C}_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$.

4) Show that the space $\mathcal{C}_c^\infty(\mathbb{R}^d)$ is dense in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

5) Let X be a Hausdorff topological space and S be a sequence of points of X. Recall the following definition:

A point x of X is said to be an accumulation point of S if every neighborhood of x contains a point of S different from x.

and show that if x is an accumulation point of the sequence S then x is an accumulation point of the filter $\mathcal{F}_S$ associated with S, where $\mathcal{F}_S := \{A \subset X : |S \setminus A| < \infty\}$.

6) Show the following statements:

a) If a Cauchy filter $\mathcal{F}$ on a t.v.s. E has an accumulation point x, then $\mathcal{F}$ converges to x.

b) A converging sequence in a Hausdorff t.v.s. E (without its limit point) is a relatively compact set in E.

c) The union of a converging sequence in a Hausdorff t.v.s. E and of its limit is a bounded set in E.