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TOPOLOGICAL VECTOR SPACES II–WS 2017/2018

Exercise Sheet 6

This exercise sheet aims to assess your progress and to explicitly work out more details of some of the results proposed in the previous lectures. Please, hand in your solutions in postbox 13 near F411 by Friday the 2nd of February at noon. The solutions to this assignment will be discussed in the ***tutorial*** *on* ***Wednesday*** *the* ***7th of February*** *(13:30–15:00) in D404.*

1) Given a family Σ of bounded subsets of a t.v.s. E such that

(P1) If $A, B \in \Sigma$, then there exists $C \in \Sigma$ such that $A \cup B \subseteq C$.

(P2) If $A \in \Sigma$ and $\lambda \in \mathbb{K}$, then there exits $B \in \Sigma$ such that $\lambda A \subseteq B$.

we define for any $A \in \Sigma$ and $\varepsilon > 0$ the following subset of E':

$$W_\varepsilon(A) := \left\{ x' \in E' : \sup_{x \in A} |\langle x', x \rangle| \leq \varepsilon \right\}.$$

Show that the family $\mathcal{B} := \{W_\varepsilon(A) : A \in \Sigma, \varepsilon > 0\}$ is a basis of neighbourhoods of the origin for the $\Sigma-$topology on E'.

2) Given a t.v.s. E, show that a sequence $\{x'_n\}_{n\in\mathbb{N}}$ of elements in E' converges to the origin in the weak topology if and only if at each point $x \in E$ the sequence of their values $\{\langle x'_n, x \rangle\}_{n\in\mathbb{N}}$ converges to zero in $\mathbb{K}$.

3) Given a t.v.s. E, show that the weak topology on E' is the coarsest topology on E' such that for all $x \in E$ the map

$$\begin{array}{rccc} v_x : & E' & \to & \mathbb{K} \\ & x' & \mapsto & \langle x', x \rangle \end{array}$$

is continuous.

4) Let $0 < p < 1$ and fix some $a, b \in \mathbb{R}$ with $a < b$. Consider the space $L^p([a,b])$ of all measurable functions $f : [a,b] \to \mathbb{R}$ such that $\int_a^b |f(t)|^p \, \mathrm{d}t < \infty$. Define a map

$$q_p(f) := \left(\int_a^b |f(t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}} \quad \text{for all } f \in L^p([a,b])$$

and set $U(\varepsilon) := \{f \in L^p([a,b]) : q_p(f) \leq \varepsilon\}$ for $\varepsilon > 0$. Show that

a) The sets $U(\varepsilon)$ with $\varepsilon > 0$ form a basis of neighbourhoods of the origin for a topology τ compatible with the vector space structure of $L^p([a,b])$.

Hint: Use the inequality $q_p(f+g) \leq 2^{\frac{1-p}{p}} (q_p(f) + q_p(g))$ for $f, g \in L^p([a,b])$.

b) The topological dual of $(L^p([a,b]), \tau)$ consists only of the zero functional.