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Exercise Sheet 7 - Solution

1) Given two sets X and Y, let E (resp. F) be the linear space of all functions from X (resp. Y) to $\mathbb{K}$ endowed with the usual addition and multiplication by scalars. For any $f \in E$ and $g \in F$, define:

$$\begin{array}{rccc} f \otimes g: & X \times Y & \to & \mathbb{K} \\ & (x,y) & \mapsto & f(x)g(y). \end{array}$$

Show that $E \otimes F := \text{span}\{f \otimes g : f \in E, g \in F\}$ is a tensor product of E and F.

Proof. Define the map $\phi : E \times F \to E \otimes F$ as $\phi(f,g) := f \otimes g$. Since $E \otimes F = \text{span}\{\phi(E \otimes F)\}$, to prove that $E \otimes F$ is a tensor product of E and F, we need only to show that ϕ is bilinear and that E and F are ϕ-linearly disjoint.

Let $\lambda \in \mathbb{K}, f, g \in E$ and $h \in F$. Then, for all $(x,y) \in X \times Y$ we have

$$\begin{aligned} ((f + \lambda g) \otimes h)(x,y) &= (f + \lambda g)(x)h(y) \\ &= f(x)h(y) + \lambda g(x)h(y) \\ &= (f \otimes h)(x,y) + \lambda(g \otimes h)(x,y), \end{aligned}$$

i.e. $\phi(f + \lambda g, h) = \phi(f,h) + \lambda\phi(g,h)$. This proves the linearity of ϕ in its first argument. The linearity in the second argument can be proved analogously.

Let $\{f_1, \dots, f_r\} \subseteq E$ and $\{g_1, \dots, g_r\} \subseteq F$ such that $\sum_{i=1}^r f_i \otimes g_i = 0$. If $\{f_1, \dots, f_r\}$ is linearly independent, then

$$0 = \sum_{i=1}^r (f_i \otimes g_i)(x,y) = \sum_{i=1}^r f_i(x)g_i(y) \text{ for all } x \in X, y \in Y.$$

By the linear independence of $\{f_1, \dots, f_r\}$, this yields $g_1(y) = \dots = g_r(y) = 0$ for all $y \in Y$. Hence, we obtain $g_1 = \dots = g_r = 0$. If $\{g_1, \dots, g_r\}$ is assumed to be linearly independent, then one can show that $f_1 = \dots = f_r = 0$ arguing similarly. □

2) Given $n, m \in \mathbb{N}$, let X and Y be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Using the approximation results in Section 1.5 in the lecture notes, prove that $\mathcal{C}_c^\infty(X) \otimes \mathcal{C}_c^\infty(Y)$ is sequentially dense in $\mathcal{C}_c^\infty(X \times Y)$.

Proof. By Exercise 1) we can form the tensor product $\mathcal{C}_c^\infty(X)\otimes\mathcal{C}_c^\infty(Y)=\mathrm{span}\{f\otimes g: f\in\mathcal{C}_c^\infty(X), g\in\mathcal{C}_c^\infty(Y)\}$ with $f\otimes g(x,y)=f(x)g(y)$ for all $x\in X, y\in Y$. By Corollary 1.5.9, polynomials in $\mathbb{R}[\underline{x},\underline{y}]$ with tuples of variables $\underline{x}=(x_1,\dots,x_n)$ and $\underline{y}=(y_1,\dots,y_m)$ are sequentially dense in $\mathcal{C}_c^\infty(X\times Y)$ w.r.t. the $\mathcal{C}^\infty$–topology, i.e.

$$\forall\,\varphi\in\mathcal{C}_c^\infty(X\times Y)\ \exists\,(p_k)_{k\in\mathbb{N}}\subseteq\mathbb{R}[\underline{x},\underline{y}]: p_k\to\varphi \text{ as } k\to\infty \tag{1}$$

in the $\mathcal{C}^\infty$–topology. Let $\varphi\in\mathcal{C}_c^\infty(X\times Y)$ and let $K\subseteq X\times Y$ be compact such that $\mathrm{supp}(\varphi)\subseteq K$ and further, let $K_1\subseteq X$ and $K_2\subseteq Y$ be compact such that $K\subseteq K_1\times K_2$. Let $g\in\mathcal{C}_c^\infty(X)$ and $h\in\mathcal{C}_c^\infty(Y)$ such that $g(x)=1=h(y)$ for all $x\in K_1$ and all $y\in K_2$. Then

$$(g\otimes h)(x,y)=1 \text{ for all } (x,y)\in\mathrm{supp}(\varphi). \tag{2}$$

Now let us observe that for any $p\in\mathbb{R}[\underline{x},\underline{y}]$ we have

$$p(\underline{x},\underline{y})=\sum_{\alpha,\beta}c_{\alpha,\beta}\underline{x}^\alpha\underline{y}^\beta=\sum_{\alpha,\beta}c_{\alpha,\beta}(\underline{x}^\alpha\otimes\underline{y}^\beta), \forall\underline{x}\in X,\underline{y}\in Y,$$

and so $p\in\mathcal{C}_c^\infty(X)\otimes\mathcal{C}_c^\infty(Y)$. Hence, for all $k\in\mathbb{N}$ we have that: $(g\otimes h)p_k\in\mathcal{C}_c^\infty(X)\otimes\mathcal{C}_c^\infty(Y)$ for all $k\in\mathbb{N}$. Then (1) implies that $(g\otimes h)p_k\to(g\otimes h)\varphi$ in the $\mathcal{C}^\infty$–topology as $k\to\infty$ and (2) ensures that $(g\otimes h)\varphi=\varphi$, which yields the assertion. □

Let E and F be two locally convex t.v.s. over the field $\mathbb{K}$. Denote by $E\otimes_\pi F$ the tensor product $E\otimes F$ endowed with the $\pi-$topology. Prove the following statements.

3) If $\mathcal{P}$ (resp. $\mathcal{Q}$) is a family of seminorms generating the topology on E (resp. on F), then the $\pi-$topology on $E\otimes F$ is generated by the family

$$\mathcal{P}\otimes\mathcal{Q}:=\{p\otimes q:\ p\in\mathcal{P}, q\in\mathcal{Q}\},$$

where for any $p\in\mathcal{P}, q\in\mathcal{Q}, \theta\in E\otimes F$ we define:

$$(p\otimes q)(\theta):=\inf\{\rho>0:\ \theta\in\rho W\}$$

with $W:=\mathrm{conv}_b(U_p\otimes V_q)$, $U_p:=\{x\in E: p(x)\leq 1\}$ and $V_q:=\{y\in F: q(y)\leq 1\}$.

Proof. Let us preliminarily show that for any seminorms p on E and q on F we have

$$U_{p\otimes q}=\mathrm{conv}_b(U_p\otimes U_q). \tag{3}$$

Let p and q be seminorms on E and F respectively and $W:=\mathrm{conv}_b(U_p\otimes U_q)$. If $\theta\in U_{p\otimes q}$, then $(p\otimes q)(\theta)\leq 1$, i.e. for any $\varepsilon>0$ there is $\rho>0$ such that $\theta\in\rho W$ and

$$\rho<(p\otimes q)(\theta)+\varepsilon\leq 1+\varepsilon.$$

Since W is balanced by definition, this yields that $\theta\in\rho W\subseteq(1+\varepsilon)W$. Hence, by the arbitrarity of ε, we have $\theta\in W$. The inclusion $W\subseteq U_{p\otimes q}$ directly follows from the definitions.

W.l.o.g. we may assume that the families $\mathcal{P}$ and $\mathcal{Q}$ are directed. Therefore, $\mathcal{B}_{\mathcal{P}} = \{\varepsilon U_p : p \in \mathcal{P}, \varepsilon > 0\}$ (resp. $\mathcal{B}_{\mathcal{Q}} = \{\varepsilon U_q : q \in \mathcal{Q}, \varepsilon > 0\}$) is a basis of neighbourhoods of the origin in E (resp. F) see (4.5) in Section 4.2 in TVS–I. Then

$$\mathcal{B} := \{\mathrm{conv}_b(\delta U_p \otimes \varepsilon U_q) : p \in \mathcal{P}, q \in \mathcal{Q}, \delta > 0, \varepsilon > 0\}$$

is a basis of neighbourhoods of the origin in the π–topology on $E \otimes F$.

Now for any $p \in \mathcal{P}, q \in Q, \delta > 0, \varepsilon > 0$ we have

$$\mathrm{conv}_b(\delta U_p \otimes \varepsilon U_q) = \mathrm{conv}_b(U_{\delta^{-1}p} \otimes U_{\varepsilon^{-1}q}) \overset{(3)}{=} U_{\delta^{-1}p \otimes \varepsilon^{-1}q} = U_{(\delta\varepsilon)^{-1}p\otimes q} = (\delta\varepsilon) U_{p\otimes q},$$

where the equality before the last follows from Theorem 4.2.3. Then

$$\mathcal{B} \equiv \{\lambda U_{p\otimes q} : p \in \mathcal{P}, q \in \mathcal{Q}, \lambda > 0\}.$$

Since $\mathcal{P}$ and $\mathcal{Q}$ are directed, we get that $\mathcal{P} \otimes \mathcal{Q}$ is also directed because by Theorem 4.2.3 we have that

$$\max\{p_1 \otimes q_1, p_2 \otimes q_2\} = (\max\{p_1, p_2\}) \otimes (\max\{q_1, q_2\})$$

holds for all $p_1, p_2 \in \mathcal{P}, q_1, q_2 \in \mathcal{Q}$. Hence, $\mathcal{B}$ is a basis of neighbourhoods of the origin of the topology on $E \otimes F$ induced by the (directed) family $\mathcal{P} \otimes \mathcal{Q}$ and the assertion is immediate. □

4) $E \otimes_\pi F$ is Hausdorff if and only if E and F are both Hausdorff.

Proof. Let $\mathcal{P}$ (resp. $\mathcal{Q}$) be a family of seminorms generating the topology on E (resp. F) and let $\mathcal{P} \otimes \mathcal{Q}$ be defined as in Exercise 3). Then by Proposition 4.3.3 from TVS–I the spaces E, F and $E \otimes_\pi F$ are Hausdorff if and only if $\mathcal{P}, \mathcal{Q}$ and $\mathcal{P} \otimes \mathcal{Q}$ are separating.

Assume that $E \otimes_\pi F$ is Hausdorff and so $\mathcal{P} \otimes \mathcal{Q}$ is separating. Let $x \in E \setminus \{o\}, y \in F \setminus \{o\}$. Then $x \otimes y \neq o \in E \otimes F$ and hence, there are $p \in \mathcal{P}, q \in \mathcal{Q}$ such that

$$0 \neq (p \otimes q)(x \otimes y) = p(x)q(y),$$

where the last equality is due to Theorem 4.2.3. Thus, $p(x) \neq 0$ and $q(y) \neq 0$, which imply that the families $\mathcal{P}$ and $\mathcal{Q}$ are both separating and so E and F are both Hausdorff.

Conversely, assume that E and F are Hausdorff. Let $o \neq \theta \in E \otimes F$, say

$$\theta = \sum_{k=1}^{r} x_k \otimes y_k,$$

where $x_1, \ldots, x_r \in E$ (resp. $y_1, \ldots, y_r \in F$) can be assumed to be linearly independent. Since E and F are Hausdorff, by Proposition 3.2.7 (a consequence of the Hahn-Banach Theorem), there are $x' \in E'$ and $y' \in F'$ such that

$$\langle x', x_1 \rangle = \langle y', y_1 \rangle = 1 \quad \text{and} \quad \langle x', x_k \rangle = \langle y', y_k \rangle = 0 \text{ for } k \geq 2.$$

Then the linear map

$$\begin{array}{rlcl} \theta' : & E \otimes F & \to & \mathbb{K} \\ & \sum_{l=1}^{s} \xi_l \otimes \eta_l & \mapsto & \sum_{l=1}^{r} \langle x', \xi_l \rangle \langle y', \eta_l \rangle \end{array}$$

is continuous for the π–topology and satisfies $\langle \theta', \theta \rangle = 1$ by construction. In particular, θ' is $(p \otimes q)$–continuous for some $p \in \mathcal{P}$ and $q \in \mathcal{Q}$, i.e. there is some $C > 0$ such that

$$0 < 1 = \big| \langle \theta', \theta \rangle \big| \leq C(p \otimes q)(\theta).$$

This yields $(p \otimes q)(\theta) \neq 0$. Thus, the family $\mathcal{P} \otimes \mathcal{Q}$ is separating and the claim follows. $\square$