Universität Konstanz
Fachbereich Mathematik und Statistik
Dr. Maria Infusino
Patrick Michalski

TOPOLOGICAL VECTOR SPACES II–WS 2017/2018

Exercise Sheet 7

This exercise sheet aims to assess your progress and to explicitly work out more details of some of the results proposed in the previous lectures. Please, hand in your solutions in postbox 13 near F411 by Friday the 16th of February at noon. Solutions to this assignment will be made available online on Maria's webpage.

1) Given two sets X and Y, let E (resp. F) be the linear space of all functions from X (resp. Y) to $\mathbb{K}$ endowed with the usual addition and multiplication by scalars. For any $f \in E$ and $g \in F$, define:

$$\begin{array}{rccc} f \otimes g: & X \times Y & \to & \mathbb{K} \\ & (x, y) & \mapsto & f(x)g(y). \end{array}$$

Show that $E \otimes F := \text{span}\{f \otimes g : f \in E, g \in F\}$ is a tensor product of E and F.

2) Given $n, m \in \mathbb{N}$, let X and Y open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Using the approximation results in Section 1.5 in the lecture notes prove that $\mathcal{C}_c^\infty(X) \otimes \mathcal{C}_c^\infty(Y)$ is sequentially dense in $\mathcal{C}_c^\infty(X \times Y)$.

Let E and F be two locally convex t.v.s. over the field $\mathbb{K}$. Denote by $E \otimes_\pi F$ the tensor product $E \otimes F$ endowed with the $\pi-$topology. Prove the following statements.

3) If $\mathcal{P}$ (resp. $\mathcal{Q}$) is a family of seminorms generating the topology on E (resp. on F), then the $\pi-$topology on $E \otimes F$ is generated by the family

$$\{p \otimes q : \, p \in \mathcal{P}, q \in \mathcal{Q}\},$$

where for any $p \in \mathcal{P}, q \in \mathcal{Q}, \theta \in E \otimes F$ we define:

$$(p \otimes q)(\theta) := \inf\{\rho > 0 : \, \theta \in \rho W\}$$

with $W := \text{conv}_b(U_p \otimes V_q)$, $U_p := \{x \in E : p(x) \leq 1\}$ and $V_q := \{y \in F : q(y) \leq 1\}$.

4) $E \otimes_\pi F$ is Hausdorff if and only if E and F are both Hausdorff.