The primary source for these notes is [7] and [4]. However, often we also took inspiration from [5] and [6].

Introduction

The theory of topological vector spaces (TVS), as the name suggests, is a beautiful connection between topological and algebraic structures. It has its origin in the need of extending beyond the boundaries of Hilbert and Banach space theory to catch larger classes of spaces and so to better understand their common features eliminating the contest-specific clutter and exploring instead the power of the general structure behind them. The first systematic treatment of these spaces appeared in "Livre V: Espaces vectoriels topologiques (1953)" in the series "Éléments de mathématique" by Nicolas Bourbaki. Actually, there was no person called Nicolas Bourbaki but this was just a pseudonym under which a group of mathematicians wrote the above mentioned series of books between 1935 and 1983 with the aim of reformulating the whole mathematics on an extremely formal, rigourous and general basis grounded on set theory. The work of the Bourbaki group (officially known as the "Association of collaborators of Nicolas Bourbaki") greatly influenced the mathematic world and led to the discovery of concepts and terminologies still used today (e.g. the symbol $\emptyset$, the notions of injective, surjective, bijective, etc.) The Bourbaki group included several mathematicians connected to the École Normale Supérieure in Paris such as Henri Cartan, Jean Coulomb, Jean Dieudonné, André Weil, Laurent Schwartz, Jean-Pierre Serre, Alexander Grothendieck. The latter is surely the name which is most associated to the theory of TVS. Of course great contributions to this theory were already given before him (e.g. the Banach and Hilbert spaces are examples of TVS), but Alexander Grothendieck was engaged in a completely general approach to the study of these spaces between 1950 and 1955 (see e.g. [1, 2]) and collected some among the deepest results on TVS in his Phd thesis [3] written under the supervision of Jean Dieudonné and Laurent Schwartz. After his dissertation he said: "There is nothing more to do, the subject is dead". Despite this sentence come out of the mouth of a genius, the theory of TVS is far from being dead. Many aspects are in fact still unknown and the theory lively interacts with several interesting problems which are still currently unsolved!

Chapter 1

Preliminaries

1.1 Topological spaces

1.1.1 The notion of topological space

The topology on a set X is usually defined by specifying its open subsets of X. However, in dealing with topological vector spaces, it is often more convenient to define a topology by specifying what the neighbourhoods of each point are.

Definition 1.1.1. *A* topology τ *on a set* X *is a family of subsets of* X *which satisfies the following conditions:*
(O1) *the empty set* $\emptyset$ *and the whole* X *are both in* τ
(O2) τ *is closed under finite intersections*
(O3) τ *is closed under arbitrary unions*
The pair (X, τ) *is called a* topological space.

The sets $O \in \tau$ are called *open sets* of X and their complements $C = X \setminus O$ are called *closed sets* of X. A subset of X may be neither closed nor open, either closed or open, or both. A set that is both closed and open is called a *clopen set*.

Definition 1.1.2. *Let* (X, τ) *be a topological space.*
- *A subfamily* $\mathcal{B}$ *of* τ *is called a* basis *if every open set can be written as a union (possibly empty) of sets in* $\mathcal{B}$.
- *A subfamily* $\mathcal{X}$ *of* τ *is called a* subbasis *if the finite intersections of its sets form a basis, i.e. every open set can be written as a union of finite intersections of sets in* $\mathcal{X}$.

Therefore, a topology τ on X is completely determined by a basis or a subbasis.

Examples 1.1.3.

a) *The family* $\mathcal{B} := \{(a, b) : a, b \in \mathbb{Q}$ *with* $a < b\}$ *is a basis of the euclidean (or standard) topology on* $\mathbb{R}$.

b) *The collection* $\mathcal{S}$ *of all semi-infinite intervals of the real line of the forms* $(-\infty, a)$ *and* $(a, +\infty)$, *where* $a \in \mathbb{R}$ *is not a basis for any topology on* $\mathbb{R}$. *To show this, suppose it were. Then, for example,* $(-\infty, 1)$ *and* $(0, \infty)$ *would be in the topology generated by* S, *being unions of a single basis element, and so their intersection* $(0, 1)$ *would be by the axiom (O2) of topology. But* $(0, 1)$ *clearly cannot be written as a union of elements in* $\mathcal{S}$. *However,* $\mathcal{S}$ *is a subbasis of the euclidean topology on* $\mathbb{R}$.

Proposition 1.1.4. *Let* X *be a set and let* $\mathcal{B}$ *be a collection of subsets of* X. $\mathcal{B}$ *is a basis for a topology* τ *on* X *iff the following hold:*

1. $\mathcal{B}$ *covers* X, *i.e.* $\forall\, x \in X, \exists\, B \in \mathcal{B}$ *s.t.* $x \in B$.
2. *If* $x \in B_1 \cap B_2$ *for some* $B_1, B_2 \in \mathcal{B}$, *then* $\exists\, B_3 \in \mathcal{B}$ *s.t.* $x \in B_3 \subseteq B_1 \cap B_2$.

Proof. (Recap Sheet 1) □

Definition 1.1.5. *Let* (X, τ) *be a topological space and* $x \in X$. *A subset* U *of* X *is called a* neighbourhood *of* x *if it contains an open set containing the point* x, *i.e.* $\exists\, O \in \tau$ *s.t.* $x \in O \subseteq U$. *The family of all neighbourhoods of a point* $x \in X$ *is denoted by* $\mathcal{F}_\tau(x)$. *(In the following, we will omit the subscript whenever there is no ambiguity on the chosen topology.)*

In order to define a topology on a set by the family of neighbourhoods of each of its points, it is convenient to introduce the notion of filter. Note that the notion of filter is given on a set which does not need to carry any other structure. Thus this notion is perfectly independent of the topology.

Definition 1.1.6. *A* filter *on a set* X *is a family* $\mathcal{F}$ *of subsets of* X *which fulfills the following conditions:*

(F1) *the empty set* $\emptyset$ *does not belong to* $\mathcal{F}$

(F2) $\mathcal{F}$ *is closed under finite intersections*

(F3) *any subset of* X *containing a set in* $\mathcal{F}$ *belongs to* $\mathcal{F}$

Definition 1.1.7. *A family* $\mathcal{B}$ *of non-empty subsets of a set* X *is a* basis of a filter $\mathcal{F}$ *on* X *if*

1. $\mathcal{B} \subseteq \mathcal{F}$
2. $\forall A \in \mathcal{F}, \exists\, B \in \mathcal{B}$ *s.t.* $B \subseteq A$

Examples 1.1.8.

a) *The family* $\mathcal{G}$ *of all subsets of a set* X *containing a fixed non-empty subset* S *is a filter and* $\mathcal{B} = \{S\}$ *is its basis.* $\mathcal{G}$ *is called the* principal filter *generated by* S.

b) Given a topological space X and $x \in X$, the family $\mathcal{F}(x)$ is a filter.
c) Let $S := \{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in a set X. Then the family $\mathcal{F}_S := \{A \subset X : |S \setminus A| < \infty\}$ is a filter and it is known as the filter associated *to S. For each $m \in \mathbb{N}$, set $S_m := \{x_n \in S : n \geq m\}$. Then $\mathcal{B} := \{S_m : m \in \mathbb{N}\}$ is a basis for $\mathcal{F}_S$.*

Proof. (Recap Sheet 1). □

Proposition 1.1.9. *A family $\mathcal{B}$ of non-empty subsets of a set X is a basis of a filter on X if and only*

$$\forall\ B_1, B_2 \in \mathcal{B}, \exists B_3 \in \mathcal{B} \text{ s.t. } B_3 \subseteq B_1 \cap B_2. \tag{1.1}$$

Proof.
$\Rightarrow$ Suppose that $\mathcal{B}$ is a basis of a filter $\mathcal{F}$ on X and let $B_1, B_2 \in \mathcal{B}$. Then, by Definition 1.1.7-1 and (F2), we get $B_1, B_2 \in \mathcal{F}$ and so $B_1 \cap B_2 \in \mathcal{F}$. Hence, by Definition 1.1.7-2, there exists $B_3 \in \mathcal{B}$ s.t. $B_3 \subseteq B_1 \cap B_2$, i.e. (1.1) holds.
$\Leftarrow$ Suppose that $\mathcal{B}$ fulfills (1.1). Then

$$\mathcal{F}_{\mathcal{B}} := \{A \subseteq X : A \supseteq B \text{ for some } B \in \mathcal{B}\} \tag{1.2}$$

is a filter on X (often called the *filter generated by* $\mathcal{B}$). In fact, (F1) and (F3) both directly follow from the definition of $\mathcal{F}_{\mathcal{B}}$ and (F2) holds, because for any $A_1, A_2 \in \mathcal{F}_{\mathcal{B}}$ there exist $B_1, B_2 \in \mathcal{B}$ such that $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$, and hence (1.1) provides the existence of $B_3 \in \mathcal{B}$ such that $B_3 \subseteq B_1 \cap B_2 \subseteq A_1 \cap A_2$, which yields $A_1 \cap A_2 \in \mathcal{F}_{\mathcal{B}}$. It is totally clear from the definition of $\mathcal{F}_{\mathcal{B}}$ that Definition 1.1.7 is fulfilled and so that $\mathcal{B}$ is basis for the filter $\mathcal{F}_{\mathcal{B}}$. □

Theorem 1.1.10. *Given a topological space X and a point $x \in X$, the filter of neighbourhoods $\mathcal{F}(x)$ satisfies the following properties.*
(N1) *For any $A \in \mathcal{F}(x)$, $x \in A$.*
(N2) *For any $A \in \mathcal{F}(x)$, $\exists\, B \in \mathcal{F}(x)$: $\forall\, y \in B$, $A \in \mathcal{F}(y)$.*
Viceversa, if for each point x in a set X we are given a filter $\mathcal{F}_x$ fulfilling the properties (N1) and (N2) then there exists a unique topology τ s.t. for each $x \in X$, $\mathcal{F}_x$ is the family of neighbourhoods of x, i.e. $\mathcal{F}_x \equiv \mathcal{F}(x)$, $\forall x \in X$.

This means that a topology on a set is uniquely determined by the family of neighbourhoods of each of its points.

Proof.
$\Rightarrow$ Let (X, τ) be a topological space, $x \in X$ and $\mathcal{F}(x)$ the filter of neighbourhoods of x. Then (N1) trivially holds by definition of neighbourhood of x. To show (N2), let us take $A \in \mathcal{F}(x)$. By the definition of neighbourhood of x,

we know that there exists $B \in \tau$ s.t. $x \in B \subseteq A$ and also that $B \in \mathcal{F}(x)$. Moreover, since for any $y \in B$ we have that $y \in B \subseteq A$ and $B \in \tau$, we can conclude that $A \in \mathcal{F}(y)$.
$\Leftarrow$ Suppose that for any x in a set X we have a filter $\mathcal{F}_x$ fulfilling (N1) and (N2). We aim to show that $\tau := \{O \subseteq X : \forall\, x \in O,\ O \in \mathcal{F}_x\}$[1] is the unique topology such that $\mathcal{F}_x \equiv \mathcal{F}_\tau(x)$, $\forall x \in X$.
Let us first prove that τ is a topology.

- $\emptyset \in \tau$ by definition of τ. Also $X \in \tau$, because for any $x \in X$ and any $A \in \mathcal{F}_x$ we clearly have $X \supseteq A$ and so by (F3) $X \in \mathcal{F}_x$.
- For any $O_1, O_2 \in \tau$, either $O_1 \cap O_2 = \emptyset \in \tau$ or there exists $x \in O_1 \cap O_2$. In the latter case, by definition of τ, we have that $O_1 \in \mathcal{F}_x$ and $O_2 \in \mathcal{F}_x$, which imply by (F2) that $O_1 \cap O_2 \in \mathcal{F}_x$ and so $O_1 \cap O_2 \in \tau$.
- Let U be an arbitrary union of sets $U_i \in \tau$. If U is empty then $U \in \tau$, otherwise let $x \in U$. Then there exists at least one i s.t. $x \in U_i$ and so $U_i \in \mathcal{F}_x$ because $U_i \in \tau$. But $U \supseteq U_i$, then by (F3) we get that $U \in \mathcal{F}_x$ and so $U \in \tau$.

It remains to show that τ on X is actually s.t. $\mathcal{F}_x \equiv \mathcal{F}_\tau(x)$, $\forall x \in X$.

- Any $U \in \mathcal{F}_\tau(x)$ is a neighbourhood of x and so there exists $O \in \tau$ s.t. $x \in O \subseteq U$. Then, by definition of τ, we have $O \in \mathcal{F}_x$ and so (F3) implies that $U \in \mathcal{F}_x$. Hence, $\mathcal{F}_\tau(x) \subseteq \mathcal{F}_x$.
- Let $U \in \mathcal{F}_x$ and set $W := \{y \in U : U \in \mathcal{F}_y\} \subseteq U$. Since $x \in U$ by (N1), we also have $x \in W$. Moreover, if $y \in W$ then $U \in \mathcal{F}_y$ and so (N2) implies that there exists $V \in \mathcal{F}_y$ s.t. $\forall z \in V$ we have $U \in \mathcal{F}_z$. This means that $z \in W$ and so $V \subseteq W$. Then $W \in \mathcal{F}_y$ by (F3). Hence, we have showed that if $y \in W$ then $W \in \mathcal{F}_y$, i.e. $W \in \tau$. Summing up, we have just constructed an open set W s.t. $x \in W \subseteq U$, i.e. $U \in \mathcal{F}_\tau(x)$, and so $\mathcal{F}_x \subseteq \mathcal{F}_\tau(x)$.

Note that the non-empty open subsets of any other topology τ' on X such that $\mathcal{F}_x \equiv \mathcal{F}_{\tau'}(x)$, $\forall x \in X$ must be identical to the subsets O of X for which $O \in \mathcal{F}_x$ whenever $x \in O$. Hence, $\tau' \equiv \tau$. $\square$

Remark 1.1.11. *The previous proof in particular shows that a subset is open if and only if it is a neighbourhood of each of its points.*

Definition 1.1.12. *Given a topological space X, a basis $\mathcal{B}(x)$ of the filter of neighbourhoods $\mathcal{F}(x)$ of $x \in X$ is called a* basis of neighbourhoods of x, *i.e. $\mathcal{B}(x)$ is a subset of $\mathcal{F}(x)$ s.t. every set in $\mathcal{F}(x)$ contains one in $\mathcal{B}(x)$. The elements of $\mathcal{B}(x)$ are called* basic neighbourhoods of x.

[1] Note that $\emptyset \in \tau$ since a statement that asserts that all members of the empty set have a certain property is always true (vacuous truth).

Example 1.1.13. *The open sets of a topological space other than the empty set always form a basis of neighbourhoods.*

Theorem 1.1.14. *Given a topological space X and a point $x \in X$, a basis of open neighbourhoods $\mathcal{B}(x)$ satisfies the following properties.*
(B1) *For any $U \in \mathcal{B}(x)$, $x \in U$.*
(B2) *For any $U_1, U_2 \in \mathcal{B}(x)$, $\exists\, U_3 \in \mathcal{B}(x)$ s.t. $U_3 \subseteq U_1 \cap U_2$.*
(B3) *If $y \in U \in \mathcal{B}(x)$, then $\exists\, W \in \mathcal{B}(y)$ s.t. $W \subseteq U$.*
Viceversa, if for each point x in a set X we are given a collection of subsets $\mathcal{B}_x$ fulfilling the properties (B1), (B2) and (B3) then there exists a unique topology τ s.t. for each $x \in X$, $\mathcal{B}_x$ is a basis of neighbourhoods of x, i.e. $\mathcal{B}_x \equiv \mathcal{B}(x)$, $\forall x \in X$.

Proof. The proof easily follows by using Theorem 1.1.10. □

The previous theorem gives a further way of introducing a topology on a set. Indeed, starting from a basis of neighbourhoods of X, we can define a topology on X by setting that a set is open iff whenever it contains a point it also contains a basic neighbourhood of the point. Thus a topology on a set X is uniquely determined by a basis of neighbourhoods of each of its points.